A STUDY OF ACCIDENTAL DEGENERACY IN HAMILTONIAN MECHANICS

By

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INTRODUCTION

Purpose

One ordinarily defines the accidental degeneracy of a system as that which does not follow from an analysis of the obvious geometrical symmetry of the system in configuration space. Frequently, nevertheless it can be shown that there is a higher order symmetry group in the phase space of the system, formed with the help of certain hidden symmetries, which is adequate to account for this accidental degeneracy. A well known example of a system possessing this property, and one of the first to be investigated historically, both classically and quantum mechanically, is the non-relativistic Kepler problem, whose quantum mechanical analogue is the problem of a one-electron atom.

In the quantum mechanical case, the invariance of the Hamiltonian with respect to the three dimensional rotation group accounts for the degeneracy in the "m" quantum number, but not for that in the "l" quantum number.

The purpose of this work is to review the more important results obtained on the accidental degeneracy of harmonic oscillators and the Kepler problem and to find some general principles with which the symmetry groups of Hamiltonian systems can be found. In Chapter 1 a basic mathematical foundation is laid. This foundation is based on the fact that all quadratic functions over the phase space of a system form a Lie algebra. On this basis a method is developed which enables

one to find, rather easily, the constants of the motion for systems with quadratic Hamiltonians or for those which can be made quadratic through a canonical transformation.

This method is then applied to various systems in the succeeding two chapters. The harmonic oscillator is treated in Chapter 2, both in the two and in the n dimensional case. Chapter 3 deals with a general Hamiltonian which may be interpreted as that of a charged mass point in a plane harmonic oscillator potential and uniform magnetic field. Finally, in Chapter 4 the Kepler problem is discussed and found to have an SU_3 symmetry besides the R_4 generally associated with it.

All calculations are carried out classically. The results obtained seem to be valid in the quantum mechanical case except as noted. The transition to the quantum mechanical version is carried out in the usual manner by replacing Poisson bracket relations with commutator brackets and replacing functions by their corresponding quantum mechanical operators.

Historical Sketch

W. Pauli [1] was perhaps the first to associate the accidental degeneracy of the Kepler problem with the less well known second vector constant of the motion, the Runge vector $\overline{R}$, which had been discussed earlier by W. Lenz [2]. Pauli defines this vector classically as lying in the plane of the orbit with a direction from the force center to the

aphelion and a length equal to the eccentricity of the classical orbit and he also gives the quantum mechanical operator definition. The commutation rules between the components of this vector and the angular momentum are also stated explicitly. As a consequence of the the commutation rules he shows that the system is degenerate in "l" also. However, L. Hulthen [3] observed the fact that these commutation rules are the same as those of the generators of the four dimensional rotation group. O. Klein was given credit for first recognizing this fact.

V. Fock [4], in 1935, using a stereographic projection in momentum space, solved the Schrödinger equation in momentum space and showed that the solutions are spherical functions of the four dimensional sphere and hence that the symmetry group is R_4. Later V. Bargmann [5], in a discussion of Fock's results, showed that Fock's group was generated by the components of the two vector constants $\overline{R}$ and $\overline{L}$, the angular momentum.

It was about this time that O. Laporte [6] obtained a differential operator identical to the Runge vector and presented a method for obtaining all the eigenfunctions of the hydrogen atom by differentiation, similar to the method used to find the eigenfunctions of L^2 by using the raising and lowering operators L_+ and L_-. He made extensive use of the stereographic parameters, developed by him and G.Y. Rainich [7] about the same time, in momentum space. The wave functions were found to be proportional to hyperspherical surface harmonics.

In 1939 and 1940 E.L. Hill and his student, J.M. Jauch [8, 9, 10] published two articles and a Ph.D. thesis dealing with accidental degeneracy. These authors considered both the two and three dimensional

Kepler problem and the harmonic oscillator in two and three dimensions. The Kepler problem was found to have R_3 and R_4 symmetry groups in two and three dimensions respectively, while the harmonic oscillators had the symmetry of SU_2 and SU_3, the groups being generated by the constants of the motion in all cases. The transformations induced by the generators were found to have an analogy in classical mechanics, namely that they represented transformations of one orbit of phase space into another of the same energy. They also showed the existence of a correspondence between transformations in classical mechanics and quantum mechanics.

In 1949, A.W. Saenz [11] wrote a Ph.D. thesis under Laporte, in which the symmetry groups explaining the accidental degeneracy of various problems were found by reducing the problems to force free motion on hyperspherical surfaces of various dimensions. In this manner he considered the Kepler problem, harmonic oscillator and rigid rotor.

More recently, the n dimensional isotropic oscillator has been discussed by G.A. Baker [12]. He found the synmetry group to be SU_n and demonstrated that this group could account for all the degeneracy found. This had been demonstrated earlier by Y.N. Demkov [13] from a slightly different point of view. Whereas Baker attacked the problem by finding the most general group which left the Hamiltonian invariant, Demkov used the generators of infinitesimal transformations which commuted with the Hamiltonian and then showed that this defined an SU_n group. In a later article Demkov [14] demonstrated the equivalence of his and Baker's works.

The connection between accidental degeneracy and hidden symmetry was confirmed by A.P. Alliluev [15] in 1958. He considered two examples, the two dimensional oscillator and the Kepler problem in n dimensions, which reduced to Fock's result in the case where n = 3. In the general case, the hidden symmetry of the Kepler problem in n dimensions was found to be that of the n + 1 dimensional hypersphere. The one dimensional Kepler problem has been treated by R. Loudon [16] and was found to be doubly degenerate in all levels except the ground state, an apparent violation of the theorem that the energy levels of one dimensional systems are nondegenerate. Loudon discusses his results in the light of this theorem and shows that it holds only when the potential is free of singularities. In 1959, H.V. McIntosh [17] wrote a qualitative review of the work up to that time and stated the results. The most recent article to appear is one by Demkov [18] in which the concept of excessive symmetry groups is introduced. He gives a prescription for finding the minimal symmetry group and as an example applies it to the anisotropic two dimensional oscillator with incommensurable frequencies.

In addition to the works cited above dealing with thestudy of accidental degeneracy itself, there has been a considerable amount of work in using the symmetry groups for various problems and applying them to the many body problem, particularly to the shell model of the nucleus. One of the first of these was by J.P. Elliott [19, 20] who undertook to study collective motion in the shell model utilizing a coupling scheme associated with the degeneracy of the three dimensional

harmonic oscillator. The direct result was a classification of the states according to the group SU_3. Following this treatment was a series of papers by V. Bargmann and M. Moshinsky [21, 22] on the group theory of harmonic oscillators, in which they developed a classification scheme for states of n particles in a common harmonic oscillator potential. They were primarily concerned with finding a scheme which would exhibit explicitly the collective nature of the states. Similar work was also done by S. Goshen and H.J. Lipkin [23, 24] with the exception that the particles were in a one dimensional potential well. This system could also be described from a collective viewpoint and these states could be classified using the group of transformations.

In recent times, L.C. Biedenharn has extensively studied the Kepler problem and its symmetries, both in the non-relativistic and relativistic cases. In the non-relativistic [25] case he has obtained relations for the representations of R_4 exploiting the fact that R_4 is locally isomorphic to $R_3 \times R_3$. The non-relativistic problem has also been considered by Moshinsky [26], where use has been made of the accidental degeneracy in the coulomb problem to obtain correlated wave functions for a system of n particles in this potential. In the relativistic case, Biedenharn [27] and Biedenharn and Swamy [28] have obtained operators which are analogous to the angular momentum and Runge vector in the non-relativistic problem.

CHAPTER 1

MATHEMATICAL FOUNDATIONS

Introduction

In classical and quantum mechanics "quadratic" Hamiltonians play an exceptional role, a fact which is traceable in the end to the fact that the Poisson bracket operation satisfies axioms whose expression is tantamount to saying that the Poisson bracket is an alternating bilinear functional which is a derivative in each of its arguments.

If $H^{(n)}$ designates the vector space of homogeneous polynomials of degree n, whose basis may be chosen to be composed of homogeneous monomials, the Poisson bracket operation maps the cartesian product $H^{(m)} \times H^{(n)}$ into $H^{(m+n-2)}$, as will be shown later. The Poisson bracket operation is a bilinear functional, which means that it defines a linear transformation when either argument is held fixed, and so it always has a matrix representation. For the case m = 2, one has a mapping from $H^{(n)}$ into itself, representable by a square matrix. In this latter case, the extensive lore of matrix theory is available to help discuss the transformations. In particular, the mapping may be described in terms of its eigenvalues and eigenvectors.

Thus the mapping

$$T_q(p) = \{q, p\}, \qquad (1.1)$$

for a fixed $q \in H^{(2)}$, and argument $p \in H^{(n)}$, can be expected to have

specialized properties which become all the more significant when $n = 2$. In this case the elements of $H^{(2)}$ form the basis of a Lie algebra under the Poisson bracket operation. In any event, the general mapping T_q of Eq. (1.1) yields representations of this algebra as it acts on polynomials, homogeneous of various degrees as shall be shown later for the case $n = 2$. Thus the allusion to the exceptional role of quadratic Hamiltonians refers to the expectation that the algebraic structure of the Lie algebra will have its reflection in the dynamical properties of the system which it represents.

Lie Algebra Defined by Poisson Brackets

The set of 2n linearly independent coordinates and momenta of an arbitrary physical system is said to form its phase space Φ. These variables may be paired in such a way that they can be indexed by f_i, $i = n, \ldots 1, -1, \ldots -n$, where the positive indices indicate coordinates and the negative ones the corresponding conjugate momenta. The Poisson bracket of any two functions of these variables is defined by

$$\{g,h\} = \sum_{i=1}^{n} \frac{\partial g}{\partial f_i}\frac{\partial h}{\partial f_{-i}} - \frac{\partial g}{\partial f_{-i}}\frac{\partial h}{\partial f_i} . \tag{1.2}$$

It follows from Eq. (1.2) that the Poisson bracket operation satisfies the axioms of a Lie algebra [29, 37]:

$$\{f,g\} = - \{g,f\} \text{ (alternating rule)}, \tag{1.3}$$

$$\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, h\} \text{ (linear)}, \qquad (1.4)$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \text{ (Jacobi identity)}. \qquad (1.5)$$

The Poisson bracket also acts like a derivative in that it satisfies the relation

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \ . \qquad (1.6)$$

Although all second degree polynomials form a Lie algebra, the linear monomial functions of the coordinates in phase space itself also form a linear vector space Φ, where the basis may be chosen to be the 2n f_i. Denoting the Poisson bracket operation { , } as the symplectic inner product in Φ, it is noted that this product does not satisfy the usual axioms for an inner product since it is anti-symmetric under interchange of the arguments. This difficulty may be remedied in the following manner. Using the definition of the Poisson bracket it is found that

$$\{f_i, f_j\} = \begin{matrix} 1 & i = -j > 0 \\ 0 & i \neq -j \\ -1 & i = -j < 0 \ . \end{matrix} \qquad (1.7)$$

From this it is observed that if $i > 0$, the dual vector $\bar{f}_i$ to f_i may be taken to be f_{-i}. Similarly for $i < 0$ the dual vector is taken to be $-f_{-i}$. Thus the dual basis to

$$\gamma = (f_n, \ldots f_1, f_{-1}, \ldots f_{-n}) \qquad (1.8a)$$

is then

$$\gamma^{+} = (-f_{-n}, \ldots -f_{-1}, f_1, \ldots f_n). \qquad (1.8b)$$

With these definitions the inner product may be defined as

$$(f, g) = \{\bar{f}, g\}, \qquad (1.9)$$

where f and g are linear combinations of the f_i and

$$\bar{f} = \sum_i a_i^* \bar{f}_i \ . \qquad (1.10)$$

It is evident then that

$$(f_i, f_j) = \{\bar{f}_i, f_j\} = \delta_{ij} \qquad (1.11)$$

and

$$(f, f) = \{\bar{f}, f\} = \sum_i |a_i|^2 \ . \qquad (1.12)$$

The space Φ also has the peculiar property of being self-dual in a special way, namely the mapping T_f is a linear mapping and is in a one to one correspondence to the vectors in the dual space Φ^{+}.

Considering tensor products of Φ, it is observed that $\{f, g\}$ also defines a mapping on these products. If one defines

$$H^{(n)} = \Phi \otimes \Phi \otimes \ldots \otimes \Phi = \otimes^{n} \Phi, \qquad (1.13)$$

and if $f \in H^{(m)}$ and $g \in H^{(n)}$, then $\{f, g\} \in H^{(m+n-2)}$. In particular, when $m = 2$ i.e., $f \in Q \equiv H^{(2)}$, a mapping is obtained which is representable by a square matrix.

Such a mapping may be expected to have eigenfunctions and eigenvalues. Let h be a fixed element of Q (a quadratic Hamiltonian) and let g_i and λ_i be the eigenfunctions and eigenvalues such that

$$\{h,\ g_i\} = T_h(g_i) = \lambda_i g_i, \tag{1.14}$$

where the g_i are taken to be linear i.e. $g_i \in H^{(1)}$. If h is representable as a normal matrix, the g_i form a basis for the space $H^{(1)}$, and are simply linear combinations of the f_i. In particular, since the $g_i \in \Phi \equiv H^{(1)}$, they form a basis for all the tensor spaces as well where all possible products of the g_i are taken i.e.,

$$\begin{aligned} ((g_i)) &= H^{(1)} \\ ((g_i g_j)) &= H^{(2)} \\ &\vdots \\ ((g_i \dots\ g_m)) &= H^{(n)}. \end{aligned} \tag{1.15}$$

On account of the derivative rule Eq. (1.6) the following relation is satisfied

$$\begin{aligned} \{h,\ g_i g_j\} &= g_i\{h,\ g_j\} + \{h,\ g_i\}\ g_j \\ &= (\lambda_i + \lambda_j)\ g_i g_j, \end{aligned} \tag{1.16}$$

and hence the monomials in g_i are eigenfunctions of h in every $H^{(n)}$ and form a basis even in the case of degeneracy, for all possible eigenfunctions.

In cases where the operator T_f is normal, a complete set of orthogonal eigenvectors exists which will be denoted g_i. Hence, if the

eigenfunctions are indexed such that $\bar{g}_i = g_{-i}$ then the eigenvectors of any normal operator satisfy the rule

$$\{g_i, g_j\} = \delta_{i,-j}. \tag{1.17}$$

The eigenvalues λ_i corresponding to the eigenvectors g_i of the normal operator h occur in negative pairs. This follows from a consideration of the Jacobi identity

$$\{h,\{g_{-i}, g_i\}\} + \{g_{-i}, \{g_i, h\}\} + \{g_i,\{h, g_{-i}\}\} = 0,$$

thus

$$\{h, 1\} + (-\lambda_i)\{g_{-i}, g_i\} + \lambda_{-i}\{g_i, g_{-i}\} = 0,$$

and

$$(\lambda_{-i} + \lambda_i)\{g_i, g_{-i}\} = 0, \tag{1.18}$$

and hence, $\lambda_i = -\lambda_{-i}$ since $\{g_i, g_{-i}\} \neq 0$.

Since $h \in H^{(2)}$, h may be written as

$$h = \sum_{\substack{i,j=-n \\ i \geqq j}}^{n} c_{ij} g_i g_j, \tag{1.19}$$

where the g_i are eigenfunctions of h belonging to eigenvalues λ_i and are indexed so that

$$\lambda_i = -\lambda_{-i} . \tag{1.20}$$

Assuming that the eigenvalues of h are distinct and using the identity $\{h, h\} = 0$ one has, with the aid of Eq. (1.19)

$$0 = \sum_{\substack{i,j \\ i \geq j}} C_{ij}\{h,\ g_i g_j\}$$

$$= \sum_{\substack{i,j \\ i \geq j}} C_{ij}(\lambda_i + \lambda_j)\ g_i g_j. \qquad (1.21)$$

Hence each coefficient must vanish and either $C_{ij} = 0$ or $(\lambda_i + \lambda_j) = 0$. However $\lambda_i = -\lambda_j$ only if $j = -i$. Therefore

$$h = \sum_{i=1}^{n} C_{i,-i}\ g_i g_{-i}. \qquad (1.22)$$

To further evaluate these coefficients recall that

$$\{h,\ g_k\} = \lambda_k g_k, \qquad (1.23)$$

thus

$$\lambda_k g_k = \sum_{i=1}^{n} C_{i,-i}\ \{g_i g_{-i},\ g_k\}$$

$$= \sum_{i=1}^{n} C_{i,-i}\ [g_i\{g_{-i},\ g_k\} + g_{-i}\{g_i,\ g_k\}]. \qquad (1.24)$$

Using the fact that the coefficients C_{ij} are symmetric in the indices the coefficient of g_k on the right side is

$$\lambda_k = C_{k,-k}\ \{g_{-k},\ g_k\}, \qquad (1.25)$$

and hence

$$c_{i,-i} = \frac{\lambda_i}{\{g_{-i}, g_i\}} . \tag{1.26}$$

The final conclusion is that

$$h = \sum_{i=1} \frac{\lambda_i}{\{g_{-i}, g_i\}} g_i g_{-i} . \tag{1.27}$$

Consider now the case where h may have degenerate eigenvalues. Using the general form for h, Eq. (1.19), and calculating {h, h} as before $C_{ij} = 0$ unless $\lambda_i = -\lambda_j$. However in the present instance it is possible to have $\lambda_k = \lambda_m$, $k \neq m$ and hence $\lambda_k = -\lambda_{-m}$. Hence

$$h = \sum_{i=1}^{n} c_{i,-i}\, g_i g_{-i} + \sum_{\substack{k,m=1 \\ \lambda_k=\lambda_m \\ k \neq m}}^{n} c_{k,-m}\, g_k g_{-m} . \tag{1.28}$$

Calculating once again $\{h, g_j\}$ results in the following equations:

$$\lambda_j g_j = \sum_{i=1}^{n} c_{i,-i} \{g_i g_{-i}, g_j\} + \sum_{\substack{k,m=1 \\ \lambda_k=\lambda_m \\ k \neq m}}^{n} c_{k,-m} \{g_k g_{-m}, g_j\} \tag{1.29}$$

$$= \sum_{i=1}^{n} c_{i,-i} [g_i \{g_{-i}, g_j\} + g_{-i} \{g_i, g_j\}]$$

$$+\sum_{\substack{k,m=1 \\ \lambda_k=\lambda_m \\ k \neq m}}^{n} C_{k,-m} \left[g_k\{g_{-m}, g_j\} + g_{-m}\{g_k, g_j\}\right] .$$

Consider the first term in the second sum. If $m=j$ then $k \neq m$ implies $k \neq j$ and hence $C_{k,-l}$ must be zero. Therefore, as before

$$h = \sum_{i=1}^{n} \frac{\lambda_i}{\{g_{-i}, g_i\}} g_i g_{-i} . \qquad (1.30)$$

It should also be noted that the transformation to eigenvector coordinates for h is a canonical transformation since the eigenvectors satisfy $\{g_i, g_j\} = \delta_{i,-j}$, which is a necessary and sufficient condition for g_i and g_{-i} to be canonical variables.

Consider now the canonical form of a non-normal quadratic operator h, of the Jordan canonical form. Again

$$h = \sum_{\substack{i=-n \\ i \geq j}}^{n} \sum_{j=-n}^{n} C_{ij} g_i g_j , \qquad (1.31)$$

where the g_i are the principal vectors of h and indexed such that the same orthogonality conditions hold as before. Using the property that

$$\{h, g_i\} = g_{i+1} , \qquad (1.32)$$

it is found that

$$g_{k+1} = \sum_{\substack{i=-n \\ i \geq j}}^{n} \sum_{j=-1}^{n} C_{ij} \{g_i g_j, g_k\}$$

$$= \sum_{\substack{i=-n \\ i \geq j}}^{n} \sum_{j=-n}^{n} C_{ij} [g_i \{g_j, g_k\} + g_j \{g_i, g_k\}]$$

$$= \begin{cases} C_{k+1,-k} \, g_{k+1} & k > 0 \\ C_{-k,k+1} \, g_{k+1} & k < 0 \; . \end{cases} \tag{1.33}$$

Hence all $C_{ij} = \delta_{i,-j+1}$ and

$$h = \sum_{i=1}^{n} g_i g_{-i+1} \; . \tag{1.34}$$

The sympletic norm also produces certain symmetries and anti-symmetries. For example, the dual mapping $R(f_i)$, which maps the coordinates into momenta and the momenta into the negatives of the coordinates,

$$\left.\begin{aligned} R(f_i) &= f_{-i} \\ R(f_{-i}) &= -f_i \end{aligned}\right\} \quad i > 0 \tag{1.35}$$

preserves the inner product; i.e.

$$\{R(f_i), R(f_j)\} = \{f_i, f_j\} \; . \tag{1.36}$$

On the other hand the mapping $P(f_i)$ which simply exchanges coordinates and momenta

$$P(f_i) = f_{-i}, \tag{1.37}$$

or the time reversal mapping $T(f_i)$

$$\left.\begin{aligned} T(f_i) &= f_i \\ T(f_{-i}) &= -f_{-i} \end{aligned}\right\} \quad i > 0 \tag{1.38}$$

reverses the sign of the symplectic norm

$$\{P(f_i),\ P(f_j)\} = -\ \{f_i,\ f_j\}$$

$$\{T(f_i),\ T(f_j)\} = -\ \{f_i,\ f_j\}\ . \tag{1.39}$$

These operators may be expressed in terms of the eigenvectors, g_i of the normal operator h, previously defined. Recall also that the g_i form a set of canonical coordinates and momenta. Using the convention that positive indices represent coordinates and negative indices the corresponding canonical momenta, these operators have the following canonical forms:

$$R = \sum_{i=1}^{n} \frac{g_i g_i + g_{-i} g_{-i}}{2\{g_{-i},\ g_i\}} \tag{1.40}$$

$$P = \sum_{i=1}^{n} \frac{g_i g_i - g_{-i} g_{-i}}{2\{g_i,\ g_{-i}\}} \tag{1.41}$$

$$T = \sum_{i=1}^{n} \frac{g_i g_{-i}}{\{g_{-i},\ g_i\}}\ . \tag{1.42}$$

These operators have the following effects on eigenvectors of h:

$$T_T(g_k) = \{T,\ g_k\} = \begin{cases} g_k & k > 0 \\ -g_k & k < 0, \end{cases} \tag{1.43}$$

$$T_P(g_k) = \{P, g_k\} = g_{-k} \qquad \text{all } k, \tag{1.44}$$

$$T_R(g_k) = \{R,\ g_k\} = \begin{cases} g_{-k} & k > 0 \\ -g_{-k} & k < 0\ . \end{cases} \tag{1.45}$$

The composite of any two of the operations on an eigenfunction is equivalent to operating with the third operator:

$$T_P(T_T(g_k)) = \left\{\begin{matrix} g_{-k} & k > 0 \\ -g_{-k} & k < 0 \end{matrix}\right\} = T_R(g_k) \tag{1.46}$$

$$T_R(T_T(g_k)) = g_{-k} = T_P(g_k) \tag{1.47}$$

$$T_P(T_R(g_k)) = \left\{\begin{matrix} g_k & k > 0 \\ -g_k & k < 0 \end{matrix}\right\} = T_T(g_k)\,. \tag{1.48}$$

It is also interesting to note that the three operators satisfy the following Poisson bracket relations among themselves:

$$\{P,\ T\} = 2R \tag{1.49}$$

$$\{R,\ T\} = 2P \tag{1.50}$$

$$\{P,\ R\} = 2T \tag{1.51}$$

The composite of the normal operator h with each of these operators on the eigenvectors is also worth noting. It is relatively easy to show

that

$$T_h(T_T(g_k)) = T_T(T_h(g_k)), \qquad (1.52)$$

and hence the matrices representing h and T commute. However one finds that P and R anticommute with h, i.e.

$$T_P(T_h(g_k)) = - T_h(T_P(g_k)) \qquad (1.53)$$

and

$$T_R(T_h(g_k)) = - T_h(T_R(g_k)). \qquad (1.54)$$

The existence of either one of these last equations is sufficient to insure that the eigenvalues appear in negative pairs [30].

In the next two chapters the method described above will be applied to harmonic oscillator Hamiltonians in two and n dimensions and to variations on the oscillator Hamiltonian.

CHAPTER 2

THE HARMONIC OSCILLATOR

Introduction

The accidental degeneracy and symmetries of the harmonic oscillators has been discussed by several authors. A particularly thorough account has been given in the thesis by J.M. Jauch [9], in which he considers the isotropic oscillator in two, three and n dimensions, and finds the symmetry group to be SU_n, i.e., the Hamiltonian is invariant under transformations induced by the generators of the group. A short discussion of the anisotropic oscillator in two dimensions with commensurable frequencies is also given. The main results of this work are also contained in the notes of E.L. Hill [33]. More recent discussions of the n dimensional isotropic oscillator have been given by G.A. Baker [12] and Y.N. Demkov [13, 14, 18].

The Plane Isotropic Oscillator

In units such that the mass and spring constant are unity, the Hamiltonian of the plane isotropic harmonic oscillator is

$$H = (P_x^2 + P_y^2 + x^2 + y^2)/2 \ . \qquad (2.1)$$

With respect to the basis (x,y,P_x,P_y), the matrix representation of H

as an operator under the Poisson bracket relation is,

$$\underset{\sim}{H} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} . \tag{2.2}$$

This representation is found by calculating the effect of H on each member of the basis under Poisson bracket. With the convention that x is a column matrix with a 1 in the first row, y with a 1 in the second row and so on, the matrix representation of H is then easily constructed. The eigenvalues of the matrix $\underset{\sim}{H}$ are

$$\lambda = \pm\, i , \tag{2.3}$$

each root appearing twice. The normalized eigenvectors associated with $\lambda = +\, i$ are

$$a = (1/\sqrt{2})(P_x - ix) = 1/\sqrt{2} \begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad b = (1/\sqrt{2})(P_y - iy) = 1/\sqrt{2} \begin{pmatrix} 0 \\ -i \\ 0 \\ 1 \end{pmatrix} \tag{2.4}$$

and those associated with $\lambda = -\, i$ are

$$a^* = (1/\sqrt{2})(P_x + ix) = 1/\sqrt{2} \begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad b^* = (1/\sqrt{2})(P_y + iy) = 1/\sqrt{2} \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix}, \tag{2.5}$$

where the column matrices are the matrix representations of the functions on the left. One may easily verify this representation, for example

$$\{H,\ a\} = \{H,\ P_x - ix\} = i(P_x - ix) \tag{2.6}$$

while multiplying the matrix in Eq. (2.2) into the column matrix representing a also gives i times the same column matrix. The quantum mechanical analogues of these four quantities are the well known raising and lowering operators for the harmonic oscillator. With these four quantities ten linearly independent quadratic monomials can be constructed which are eigenfunctions of H and have as eigenvalues 0, + 2i or - 2i. Only four of these are of interest here, namely those with eigenvalue zero. They are the four linearly independent constants of the motion (since they commute with H under Poisson bracket) listed below:

$$aa^* = (P_x^2 + x^2)/2 \tag{2.7}$$

$$bb^* = (P_y^2 + y^2)/2 \tag{2.8}$$

$$ab^* = [(P_xP_y + xy) + i(yP_x - xP_y)]/2 \tag{2.9}$$

$$a^*b = [(P_xP_y + xy) - i(yP_x - xP_y)]/2\ . \tag{2.10}$$

For the purpose of physical interpretation it is more convenient to deal with the real and imaginary parts of these constants separately, as well as to separate the Hamiltonian from them. Accordingly the following four quantities are introduced:

$$H = aa^* + bb^* = (P_x^2 + P_y^2 + x^2 + y^2)/2 \tag{2.11}$$

$$D = aa^* - bb^* = (P_x^2 + x^2 - P_y^2 - y^2)/2 \tag{2.12}$$

$$L = i(a^*b - ab^*) = yP_x - xP_y \tag{2.13}$$

$$K = a^*b + ab^* = xy + P_xP_y \ . \tag{2.14}$$

These four constants have the following simple interpretations.

a) H, the Hamiltonian, is the total energy of the system.

b) D is the energy difference between the two coordinates.

c) L is the angular momentum of the system and the generator of rotations in the xy plane.

d) K is known as the correlation and is a peculiar feature of the harmonic oscillator. As a generator of an infinitesimal contact transformation, it generates an infinitesimal change in the eccentricity of the orbital ellipse while preserving the orientation of the semiaxes, and preserving the sum of the squares of their lengths. The energy of a harmonic oscillator depends only on the sum of the squares of the semiaxes of its orbital ellipse, which remains constant under such a transformation.

The set of functions {K, L, D} is closed under the Poisson bracket operation. Explicitly, their Poisson bracket table is

	K	L	D
K	0	2D	-2L
L	-2D	0	2K
D	2L	-2K	0

(2.15)

Aside from the factor 2 these are the Poisson bracket relations of the

generators of the three dimensional rotation group, or the three components of the angular momentum in three dimensions. Since the configuration space of the two dimensional isotropic oscillator is apparently only two dimensional and has only rotations in the xy plane as an obvious symmetry, the occurrence of the three dimensional rotation group is rather anomalous.

By performing a series of geometrical transformations on a, a^*, b and b^* one can explicitly demonstrate the spherical symmetry of the system. This series of transformations is sometimes called the Hopf mapping.

The Hopf Mapping [31]

Perhaps the best analytic representation of the Hopf mapping is obtained by introducing polar coordinates (λ, τ, σ, ρ) for the complex variables a and b. Explicitly, these are

$$a = (\lambda e^{-i\rho} \cos \tau)/\sqrt{2} \qquad (2.16)$$

$$b = (\lambda e^{-i\sigma} \sin \tau)/\sqrt{2}, \qquad (2.17)$$

where λ does not denote the eigenvalue of H as previously. The first step in the Hopf mapping is to form the ratio ω defined as the quotient

$$\omega = \lambda^2\, a/b\ . \qquad (2.18)$$

The formation of this ratio may be regarded as a gnomonic projection of the four dimensional space, regarded as having two complex dimensions,

onto a two dimensional space (Fig. 2.1), having one complex dimension. This maps the point (a,b) lying on the two dimensional complex circle of radius λ^2 into the point

$$\lambda^2 \, a/b = \lambda^2 \, e^{i(\sigma-\rho)} \cot \tau \tag{2.19}$$

on the complex line.

The second step is to regard the complex point $\omega = \lambda^2 e^{i(\sigma-\rho)} \cot \tau$ as not lying on a complex line; but rather as a point in a real two dimensional plane. One now performs an inverse stereographic projection onto the surface of a three dimensional sphere whose south pole is tangent to the plane at its origin. By choosing the origin of the three dimensional space to be at the center of the sphere, a point on the surface of the sphere will be specified by giving the radius r, the azimuth Φ and the colatitude θ. For convenience the sphere is chosen to have diameter λ^2. The azimuth is measured in a plane parallel to the original plane and hence

$$\Phi = \sigma - \rho \, . \tag{2.20}$$

Thus to determine the location of the projected point P on the sphere, only the colatitude θ is left to be determined. From Fig. 2.2 one has

$$\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cot \tau \tag{2.21}$$

but

$$\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cot \frac{\theta}{2} \tag{2.22}$$

hence

$$\theta = 2\tau \, . \tag{2.23}$$

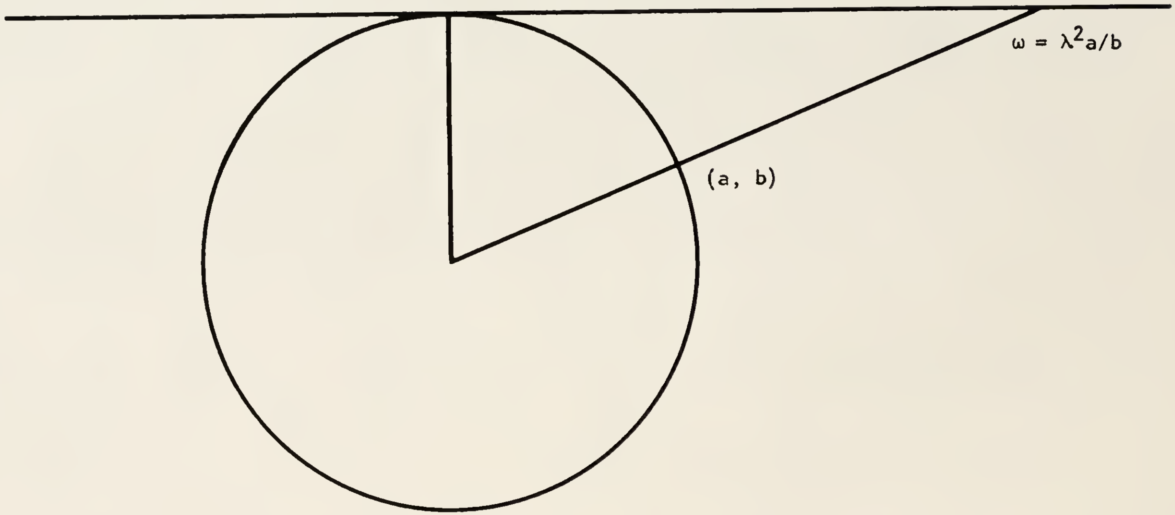


Fig. 2.1 Gnomonic Projection

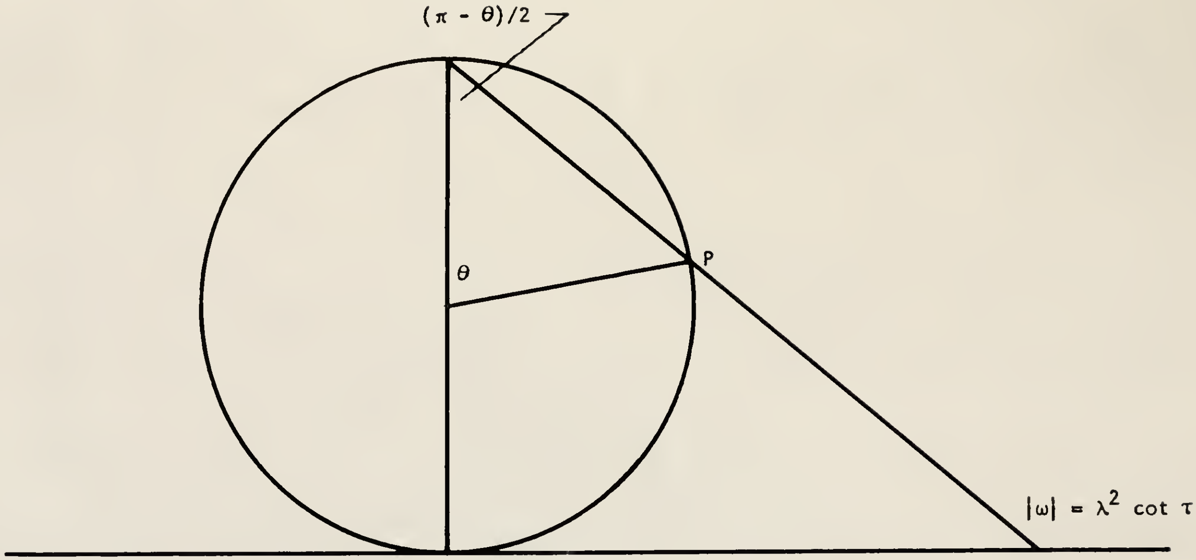


Fig. 2.2 Inverse Stereographic Projection

Thus the coordinates of the point P projected onto the surface of the sphere from the point $\omega = \lambda^2 e^{i(\sigma-\rho)} \cot \tau$ are

$$\theta = 2\tau \tag{2.24}$$

$$\phi = \sigma - \rho \tag{2.25}$$

$$r = \lambda^2/2 \ . \tag{2.26}$$

Expressing the quantities H, K, L and D in Eq. (2.4) in terms of these new coordinates gives

$$H = \lambda^2/2 \tag{2.27}$$

$$K = (\lambda^2/2) \sin \theta \cos \phi = X \tag{2.28}$$

$$L = (\lambda^2/2) \sin \theta \sin \phi = Y \tag{2.29}$$

$$D = (\lambda^2/2) \cos \theta = Z \ . \tag{2.30}$$

Hence, K, L and D simply determine a point on the surface of this sphere. It should be noted that K generates infinitesimal rotations about X, L about Y, and D about Z.

Since these three constants of the motion are then directly the coordinates of the fixed point corresponding to the orbit of the oscillator, it is apparent that they generate rotations of the sphere, which transform one orbit into another of the same energy.

Canonical Coordinates

Whenever two functions satisfy the Poisson bracket relation

$$\{A,\ B\} = \mu B \tag{2.31}$$

where μ is a scalar, a corresponding equation,

$$\{A,\ (1/\mu)\ \ln B\} = 1\ , \tag{2.32}$$

may always be written, which is the rule satisfied by a coordinate and momentum which form a canonically conjugate pair. The first of these two equations simply states that B is an eigenfunction of A with respect to the Poisson bracket operation. Consequently it is to be expected that such eigenfunctions be a ready source of canonical variables. A system in which the Hamiltonian is one of the momenta can be obtained in the present case. Consider the following diagram in which the two-headed arrows imply that the Poisson bracket of the two functions is zero and the one-headed arrows imply that the Poisson bracket is + 1 in the direction indicated:

$$\begin{tikzcd}
H = a a^* + b b^* \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & D = a a^* - b b^* \arrow[dl, leftrightarrow] \\
(1/4i)\ \ln\ (a^* b^*/ab) \arrow[u] \arrow[r, leftrightarrow] & (1/4i)\ \ln\ (ab^*/a^* b) . \arrow[u]
\end{tikzcd} \tag{2.33}$$

If these four quantities are rewritten in terms of the four parameters introduced in the Hopf mapping and $\rho + \sigma$ is denoted by Ψ, then Eq. (2.33) becomes

$$\begin{tikzcd} H = \lambda^2/2 \arrow[r, leftrightarrow] & D = (\lambda^2/2)\cos\theta \\ \Psi/2 \arrow[u] \arrow[r, leftrightarrow] \arrow[ur, leftrightarrow] & \Phi/2\ . \arrow[u] \arrow[ul, leftrightarrow] \end{tikzcd} \tag{2.34}$$

Hence the Hamiltonian and the energy difference are the two momenta and the angles $\Psi/2$ and $\Phi/2$ are the corresponding canonical coordinates. Since $\Psi/2$ is conjugate to H, Ψ increases linearly with time.

The final result of this transformation is to map an entire orbit into a single point. Since the radius of the image sphere determines the energy, the only information lost as a consequence of the mapping is the phase of the point representing the harmonic oscillator in the great circle comprising its orbit. The lost phase may be recovered by attaching a flag to the point representing the orbit, which will then rotate with constant velocity. The missing angle Ψ may then be taken as the angle which the flag makes with its local meridian. In this way the motion of the harmonic oscillator in its phase space is shown to be equivalent to the motion of a rigid spherical rotator, and in fact the parameters Φ, θ and Ψ are just the Euler angles describing the orientation of the rotor. These same parameters may also be used for the description of spinors, and a particularly lucid geometrical interpretation of this transformation may be found in a paper of Payne [32] describing various methods of representation of two-component spinors.

The Anisotropic Harmonic Oscillator in Two Dimensions

The anisotropic oscillator, both with commensurable frequencies and with non-commensurable frequencies has been discussed by Jauch and Hill [10] and also by Hill [33]. These authors were able to show that, in the case of commensurable frequencies, the symmetry group of the system is again the three dimensional rotation group. However, they were unable to come to any conclusion concerning the case with incommensurable frequencies. Recently R.L. Hudson[1] has been investigating the incommensurable case also.

It can be shown though, that even with non-commensurable frequencies the symmetry group is also a three dimensional rotation group. The proof is sketched below and follows closely that used in the isotropic case, except where noted.

If units are chosen such that the frequency in the x coordinate is unity, the Hamiltonian is

$$H = (P_x^2 + P_y^2 + x^2 + \omega^2 y^2)/2 \ , \qquad (2.35)$$

where ω is the frequency in the y coordinate and is an irrational number. The eigenfunctions and eigenvalues of H under the Poisson bracket operation are listed below:

---

[1]Private communication.

Eigenfunction	Eigenvalue
$a = (1/\sqrt{2})(P_x - ix)$	i
$a^* = (1/\sqrt{2})(P_x + ix)$	$-i$
$b = (1/\sqrt{2})(P_y - i\omega y)$	$i\omega$
$b^* = (1/\sqrt{2})(P_y + i\omega y)$	$-i\omega$.

(2.36)

Although the most obvious four constants of the motion are aa^*, bb^*, $(a)^\omega b^*$, and $(a^*)^\omega b$, a more convenient set is the following collection of independent combinations of them:

$$H = aa^* + bb^* \tag{2.37}$$

$$D = aa^* - bb^* \tag{2.38}$$

$$K = [(a)^\omega b^* + (a^*)^\omega b]/(aa^*)^{(\omega-1)/2} \tag{2.39}$$

$$L = i[(a)^\omega b^* - (a^*)^\omega b]/(aa^*)^{(\omega-1)/2} \; . \tag{2.40}$$

Again the set {K, L, D} is closed under the Poisson bracket operation and forms a Lie group, the three dimensional rotation group. The Poisson bracket table is listed below:

	K	L	D
K	0	$2\omega D$	$-2\omega L$
L	$-2\omega D$	0	$2\omega K$
D	$2\omega L$	$-2\omega K$	0 .

(2.41)

A mapping to polar coordinates as in the isotropic case can once again be performed, however it is not a Hopf mapping. The form of the final equations for the constants is the same as in Eqs. (2.27) to

(2.30), however, for the anisotropic case

$$\Phi = \omega\rho - \sigma \,. \tag{2.42}$$

A set of functions which proves to be convenient to use as the new canonical coordinates is diagrammed below with the same convention as before:

$$\begin{tikzcd}
H \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & D \arrow[dl, leftrightarrow] \\
\frac{1}{4i\omega} \ln \frac{(a^*)^{\omega} b^*}{(a)^{\omega} b} \arrow[u] \arrow[r, leftrightarrow] & \frac{1}{4i\omega} \ln \frac{(a^*)^{\omega} b}{(a)^{\omega} b^*} \arrow[u]
\end{tikzcd} \,. \tag{2.43}$$

If the parameters from the mapping are substituted into Eq. (2.43) this table becomes identical to Eq. (2.34) with the conditions that Φ is defined by Eq. (2.42) and

$$\Psi = \omega\rho + \sigma \,. \tag{2.44}$$

It is rather interesting to note that the entire effect of the possible incommensurability of the frequencies in the two coordinates is completely absorbed by the power mapping $(a)^{\omega}$, which is well defined as exp (ω ln a) in the case of an irrational ω. In this latter case one has a mapping with infinitely many branches rather than a finite number as in the case of a rational ω. It is of course on this account that the orbit--a Lissajous figure in the x-y plane--would then be open, and would not correspond to periodic motion.

Commensurability and Constants of the Motion

In reviewing the analysis of the theory of the plane harmonic oscillator, it is probably appropriate to draw attention to a number of statements made, both in the literature of classical mechanics, and the lore of accidental degeneracy. First of all, the existence of accidental degeneracy is often considered to be allied with the existence of bounded, closed orbits. When this argument is used, it is applied to the existence of algebraic integrals of the motion, and indeed it should be noted here that as long as the frequencies of the two coordinates are commensurable, bounded closed orbits result and the integrals are algebraic, as required by the theory. However, the above results also verify something which has been generally known [34], namely that non-algebraic integrals can exist, and as would be expected, are associated with space-filling motion.

Another point regards the separability of the Hamilton-Jacobi equation in several coordinate systems. It is pointed out that if a Hamilton-Jacobi equation is separable in more than one coordinate system, it is necessarily degenerate [35]. The argument usually given concerns the fact that orbits with incommensurable frequencies in one coordinate system will be space-filling curves, the boundaries of which are coordinate arcs. Thus, in whatever coordinate system the motion is described, the system of separation is uniquely defined as that one forming an envelope of the various orbits. Of course, the option left in the enunciation of this theory is the fact that changes of scale may still be made, in which the new coordinates are functions each of only

one of the old coordinates, and not several or all. So it is in the polar mapping, where one forms a ratio of powers of functions of only a single coordinate. However, this change of "scale" seems to eliminate the problem of the incommensurable frequencies at the same time, and thus circumvents the "proof" of uniqueness. This result has only been secured at a price, namely the inverse of the transformation is infinitely multiple valued [36]. Otherwise, one could easily reduce the anisotropic oscillator to a static point on the Riemann sphere, return from this to an isotropic oscillator, and separate in whatever system desired.

These considerations are not unrelated to the problem of finding a Lie group generated by the constants of the motion. In this respect, a Lie group has always been regarded as being more important quantum mechanically than classically, because its irreducible representations would be associated with accidental degeneracies of the Schrödinger equation. Since the anisotropic harmonic oscillator has no accidental degeneracies in the case of incommensurable frequencies, it is somewhat surprising to find that the unitary unimodular group is still its symmetry group, at least classically.

In the case of commensurable frequencies, proportional say to m and n, the operator $a^m b^{*n}$ commutes with the Hamiltonian, and is interpreted as creating m quanta in the first coordinate and annihilating n quanta in the second coordinate. Thus, m quanta in one coordinate are equal to n quanta in the second coordinate, the energy is restored to its original level, and the system is left in another eigenstate.

Thus, it is possible to expect and explain degeneracy in the case of commensurable frequencies.

In the classical incommensurable case, the symmetry group still exists. After all, the total energy of the oscillator is the sum of the energies in each coordinate, which is reflected in the fact that it is the sum of the squares of the semiaxes of the orbit (A Lissajous figure, like an ellipsoid, has principal axes), and not the individual semiaxes, which determine the energy. Thus, the dimensions of the bounding rectangle of a space-filling Lissajous figure may be changed subject to this constraint. This, in addition to an adjustment of the relative phases in the two coordinates, is the effect of the constants of the motion acting as the generators of infinitesimal canonical transformations.

The problem of carrying this result over to quantum mechanics seems to be in finding a proper analogue of the power functions $(a)^{\omega}$ and b, which as ladder operators may not possess fractional powers, even though they did so when considered as functions of a complex variable.

The Isotropic Oscillator in n Dimensions

Using units such that both the mass and force constant are unity, the Hamiltonian for this system is

$$H = \sum_{i=1}^{n} (p_i^2 + x_i^2)/2. \tag{2.45}$$

The eigenfunctions of H under the Poisson bracket operation are

$$a_j = (P_j - ix_j)/\sqrt{2} \tag{2.46}$$

with eigenvalues

$$\lambda = i \tag{2.47}$$

as well as

$$a_j^* = (P_j + ix_j)/\sqrt{2} \tag{2.48}$$

with eigenvalues

$$\lambda = -i\ . \tag{2.49}$$

Hence one can immediately construct n^2 linearly independent constants of the motion of the form $a_i a_j^*$, $i = 1, \ldots n$ and $j = 1, \ldots n$, although as before, it is somewhat more convenient to deal with their real and imaginary parts separately; in particular,

$$H = \sum_{i=1}^{n} a_i a_i^* \tag{2.50}$$

$$D_i = a_1 a_1^* - a_i a_i^* \qquad i = 2, \ldots n \tag{2.51}$$

$$K_{ij} = a_i a_j^* + a_i^* a_j \qquad j > i \tag{2.52}$$

$$L_{ij} = i[a_i^* a_j - a_i a_j^*] \qquad j > i\ . \tag{2.53}$$

The latter three sets of constants are closed under Poisson bracket and satisfy the following commutation relations:

$$\{K_{ij}, K_{km}\} = i[-(a_i^*a_m - a_ia_m^*)\delta_{jk} + (a_k^*a_j - a_ka_j^*)\delta_{im} - (a_i^*a_k - a_ia_k^*)\delta_{jm} + (a_m^*a_j - a_ma_j^*)\delta_{ik}] \tag{2.54}$$

$$\{L_{ij}, L_{km}\} = i[(a_i^*a_m - a_ia_m^*)\delta_{jk} + (a_j^*a_k - a_ja_k^*)\delta_{im} - (a_i^*a_k - a_ia_k^*)\delta_{jm} - (a_j^*a_m - a_ja_m^*)\delta_{ik}] \tag{2.55}$$

$$\{D_i, D_j\} = 0 \tag{2.56}$$

$$\{K_{ij}, L_{km}\} = (a_j^*a_m + a_ja_m^*)\delta_{ik} - (a_i^*a_k + a_ia_k^*)\delta_{jm} + (a_i^*a_m + a_ia_m^*)\delta_{jk} - (a_j^*a_k + a_ja_k^*)\delta_{im} \tag{2.57}$$

$$\{L_{ij}, D_k\} = (a_j^*a_k + a_ja_k^*)\delta_{ik} - (a_i^*a_k + a_ia_k^*)\delta_{jk} - (a_j^*a_1 + a_ja_1^*)\delta_{i1} \tag{2.58}$$

$$\{D_j, K_{km}\} = i[(a_j^*a_k - a_ja_k^*)\delta_{jm} + (a_j^*a_m - a_ja_m^*)\delta_{jk} - (a_1^*a_m - a_1a_m^*)\delta_{1k}] . \tag{2.59}$$

Depending upon the values of i, j, k, and m all of the relations will give linear combinations of the functions D_i, K_{ij}, or L_{ij} as a result. The relations also identify the symmetry group of the problem [37], namely SU_n where the $n^2 - 1$ constants are considered as the generators of the group.

The Hopf Mapping

In general only 2n-1 independent constants of the motion including the Hamiltonian are expected to be found and not n^2. Hence there should exist $n^2-(2n-1) = (n-1)^2$ functional relationships among the n^2 constants.

As a consequence, the following set of 2n-1 constants have been chosen for the Hopf mapping:

the (n-1) constants K_{1j} where j = 2, ...n,

the (n-1) constants L_{1j} where j = 2, ...n,

and the following linear combination of H and the D_i

$$D = B_1 H + \sum_{i=2}^{n} B_i D_i \tag{2.60}$$

where

$$B_1 = \frac{2-n}{n} \tag{2.61}$$

and

$$B_i = \frac{2}{n} \ (i = 2, \ldots n) \ . \tag{2.62}$$

In terms of the a_i and a_i^*

$$D = a_1 a_1^* - \sum_{i=2}^{n} a_i a_i^* \ . \tag{2.63}$$

The $(n-1)^2$ relationships between the constants are

$$L_{ij} = \frac{L_{1j}K_{1i} - L_{1i}K_{1j}}{H + D} \tag{2.64}$$

$$K_{ij} = \frac{L_{1i}L_{1j} + K_{1i}K_{1j}}{H + D} \tag{2.65}$$

$$D_i = \frac{(H+D)^2 - K_{1i}^2 - L_{1i}^2}{2(H + D)} \tag{2.66}$$

where

$$H^2 = D^2 + \sum_{i=2}^{n} K_{1i}^2 + \sum_{i=2}^{n} L_{1i}^2 \tag{2.67}$$

as may be verified by substitution from the definitions given in Eqs. (2.50) through (2.53).

For the Hopf mapping let the 2n variables a_i and a_i^* be redefined as

$$a_1 = \frac{\lambda}{\sqrt{2}}(\cos\tau)e^{-i\rho} \tag{2.68}$$

$$a_j = \frac{\lambda}{\sqrt{2}}\epsilon_j(\sin\tau)e^{-i\sigma_j} \quad (j = 2, \ldots n) \tag{2.69}$$

where the ϵ_j are direction cosines such that

$$\sum_{j=2}^{n} \epsilon_j^2 = 1. \tag{2.70}$$

Forming the ratios

$$\frac{a_j}{a_1} = \epsilon_j(\cot\tau)e^{i(\rho-\sigma_j)} \tag{2.71}$$

gives a complex projective space of n-1 complex dimensions. Regarding the space as having 2n-2 real dimensions and performing an inverse stereographic projection a real space of dimension 2n-1 is obtained.

As in the case of the two dimensional oscillator the sphere of 2n-1 dimensions is considered tangent to the 2n-2 dimensional space at its south pole. By defining θ as the angle between a radius vector from the center of the sphere and the diameter between the north and south poles 2τ can be identified with θ. The angles in the planes

perpendicular to the north-south diameter projected from the 2n-2 space parallel to this line are identical with the angles in the space and are denoted

$$\phi_i = \sigma_i - \rho \ . \qquad (2.72)$$

The substitution of Eqs. (2.68) and (2.69) in Eq. (2.63) and into the equations defining H, K_{1j}, and L_{1j} gives

$$H \quad = (1/2)\lambda^2 \qquad (2.73)$$

$$D \quad = (1/2)\lambda^2 \cos\theta = Z \qquad (2.74)$$

$$K_{1j} = (1/2)\lambda^2 \sin\theta\epsilon_j \cos\phi_j = X_j \qquad (2.75)$$

$$L_{1j} = (1/2)\lambda^2 \sin\theta\epsilon_j \sin\phi_j = Y_j \qquad (2.76)$$

where the K_{1i}, L_{1i}, and D have been identified with the 2n-1 coordinates in this space. As in the case for n = 2 these 2n-1 constants determine a point on the surface of a sphere of radius H in this space. At the same time the remaining constants still define infinitesimal transformations in the respective coordinate planes such as rotations or changes in the correlation or relative phases. It should be noted that only in three dimensions is it possible to identify planes and their perpendicular coordinates in a one to one fashion. In general for n dimensions there are (1/2)n(n-1) coordinate planes.

Other Canonical Coordinates

In finding canonical sets of coordinates and momenta a wide choice of momenta, even when the n momenta are selected from amongst the constants of the motion, is available. The final decision will depend upon the particular application desired. However, there are some simple linear combinations which are often sufficiently useful, to make their display worthwhile.

As a first selection for the canonical momenta consider the following:

$$P_1 = H = \sum_{i=2}^{n} a_i a_i^* \tag{2.77}$$

$$P_j = \sum_{i=1}^{j-1} a_i a_i^* - \sum_{i=j}^{n} a_i a_i^* \qquad (j = 2, \ldots n), \tag{2.78}$$

where P_j is the energy difference between the first $(j-1)$ coordinates and the last $n-(j-1)$ coordinates. The coordinates conjugate to these momenta must then have the form

$$Q_1 = \frac{1}{4i} \ln \frac{a_1^* a_n^*}{a_1 a_n} \tag{2.79}$$

$$Q_j = \frac{1}{4i} \ln \frac{a_{j-1}^* a_j^*}{a_{j-1} a_j} \qquad (j = 2, \ldots n) . \tag{2.80}$$

In terms of the parameters used in the Hopf mapping these become

$$H = \lambda^2/2 \tag{2.81}$$

$$P_2 = (\lambda^2/2)\cos\theta \tag{2.82}$$

$$P_j = (\lambda^2/2)\left[\cos^2\tau + \sum_{i=2}^{j-1}\epsilon_i^2\sin^2\tau - \sum_{i=j}^{n}\epsilon_i^2\sin^2\tau\right]$$

$$= (\lambda^2/2)\left[\cos\theta + \sum_{i=2}^{j-1}\epsilon_i^2\sin^2\theta/2\right] \quad (j = 3, \ldots n) \tag{2.83}$$

and

$$Q_1 = 1/2(\rho - \sigma_n) \tag{2.84}$$

$$Q_2 = 1/2(\rho - \sigma_2) \tag{2.85}$$

$$Q_j = 1/2(\sigma_{j-1} - \sigma_j) \quad (j = 3, \ldots n). \tag{2.86}$$

One might also consider taking the total energy and the energy in the last n-(j-1) coordinates for the canonical momenta,

$$P_1 = H = \sum_{i=1}^{n} a_i a_i^* \tag{2.87}$$

$$P_j = \sum_{i=j}^{n} a_i a_i^* \quad (j = 2, \ldots n) \tag{2.88}$$

with the corresponding coordinates

$$Q_j = \frac{1}{2i}\ln\frac{a_j^* a_{j+1}}{a_j a_{j+1}^*} \quad (j = 1, \ldots n-1) \tag{2.89}$$

$$Q_n = \frac{1}{2i}\ln\frac{a_n^*}{a_n} \tag{2.90}$$

which in terms of the Hopf parameters become

$$H = \lambda^2/2 \tag{2.91}$$

$$P_j = (\lambda^2/2)\sum_{i=j}^{n} \epsilon_i^2 \sin^2 \tau \qquad (j = 2, \ldots n) \tag{2.92}$$

$$Q_1 = \rho - \sigma_2 \tag{2.93}$$

$$Q_j = \sigma_j - \sigma_{j+1} \qquad (j = 2, \ldots n-1) \tag{2.94}$$

$$Q_n = \sigma_n \,. \tag{2.95}$$

Finally, one might consider using the energy and(n-1) successive energy differences,

$$P_1 = H\sum_{i=1}^{n} a_i a_i^* \tag{2.96}$$

$$P_j = a_{j-1}a_{j-1}^* - a_j a_j^* \qquad j = (2, \ldots n) \,. \tag{2.97}$$

The corresponding coordinates are

$$Q_1 = \frac{1}{2ni} \ln \frac{a_1^* a_2^* \ldots a_n^*}{a_1 a_2 \ldots a_n} \tag{2.98}$$

$$Q_k = \frac{1}{2ni} \ln \left(\prod_{j=1}^{k-1} \left(\frac{a_j^*}{a_j}\right)^{n-k+1} \right) \left(\prod_{j=k}^{n} \left(\frac{a_j}{a_j^*}\right)^{k-1} \right) \tag{2.99}$$

$$k = (2, \ldots n).$$

In terms of the Hopf coordinates

$$P_1 = H = \lambda^2/2 \tag{2.100}$$

$$P_2 = (\lambda^2/2)(\cos^2 \tau - \epsilon_2^2 \sin^2\tau) \tag{2.101}$$

$$P_j = (\lambda^2/2) \sin^2\tau(\epsilon_{j-1}^2 - \epsilon_j^2) \quad (j=3, \ldots n) \tag{2.102}$$

$$Q_1 = (1/n)(\rho + \sum_{i=2}^{n} \sigma_i) \tag{2.103}$$

and

$$Q_j = (1/n)[(n-j+1)(\rho + \sum_{k=2}^{j-1} \sigma_k)-(j-1) \sum_{k=j}^{n} \sigma_k] . \tag{2.104}$$

In all three cases, the transformation has been such as to give the action-angle variables. The effect has been to map the orbit into a point on the surface of a sphere of 2n-1 dimensions whose radius is the energy H. Since the mapping is many to one, one sees that as in the two dimensional case an entire orbit is mapped into a single point. The orbital phase may still be recovered by attaching a flag to the point representing the orbit which will then rotate with constant angular velocity Q_1, since Q_1 is conjugate to H and hence increases linearly with time.

The second case also has another interesting property, namely D_2 is the Hamiltonian for an isotropic oscillator in (n-1) dimensions, and the other D_i are constants of its motion. Hence one may perform another Hopf mapping and continue in this manner until one has the 2-dimensional oscillator again.

In general, any of the 2n-1 independent constants from the n^2-1 constants could be chosen as generators, and used as coordinates. It should be noted that the expressions for the coordinates could just as well be expressed in terms of an arctangent by using the well known relation

$$\tan^{-1} x = \frac{1}{2} i \ln \frac{1-ix}{1+ix} \qquad (2.105)$$

which would exhibit the similarity between these coordinates and those obtained by Goshen and Lipkin [23].

The Anisotropic Oscillator in n Dimensions

In units where the mass is unity, the Hamiltonian for the n-dimensional anisotropic harmonic oscillator is

$$H = \frac{1}{2}\sum_{i=1}^{n} (P_i^2 + \omega_i^2 x_i^2), \qquad (2.106)$$

where ω_i is the frequency in the i^{th} coordinate. The eigenvectors are all of the form

$$a_j = (P_j - i\,\omega_j\, x_j)/\sqrt{2} \qquad (2.107)$$

$$a_j^* = (P_j + i\,\omega_j\, x_j)/\sqrt{2} \qquad (2.108)$$

with eigenvalues

$$\lambda_j = \pm\, i\,\omega_j\ . \qquad (2.109)$$

There also exists n^2 constants similar to those obtained in the isotropic case, namely

$$H = \sum_{i=1}^{n} a_i a_i^* \tag{2.110}$$

$$D_i = a_i a_i - a_{i+1} a_{i+1}^* \qquad (i = 1, \ldots n-1) \tag{2.111}$$

$$K_{ij} = \frac{(a_i^{*\omega_j} a_j^{\omega_i} + a_i^{\omega_j} a_j^{*\omega_i})}{(a_i a_i^*)^{\frac{\omega_j - 1}{2}} (a_j a_j^*)^{\frac{\omega_i - 1}{2}}} \tag{2.112}$$

$$L_{ij} = \frac{i[a_i^{*\omega_j} a_j^{\omega_i} - a_i^{\omega_j} a_j^{*\omega_i}]}{(a_i a_i^*)^{\frac{\omega_j - 1}{2}} (a_j a_j^*)^{\frac{\omega_i - 1}{2}}} \;. \tag{2.113}$$

Once again this set is closed under Poisson bracket and satisfies the commutation relations of the generators of SU_n.

It should be noted that every normal quadratic Hamiltonian is equivalent to some anisotropic harmonic oscillator, because they can be put in the canonical form given in Chapter 1.

For non-normal Hamiltonians such as the free particle

$$H = \frac{1}{2m} \bar{P}^2 \;, \tag{2.114}$$

there is no equivalence with the harmonic oscillator. However, symmetry groups still exist for these cases; for example the above expression is invariant under rotations. The symmetry of such Hamiltonians has been discussed in considerable detail by Moshinsky [38] and F.T. Smith [39, 40].

For the Hamiltonian given in Eq. (2.114) (where m = 1) the following set of nine quadratic constants of the motion exists,

$$H = (P_x^2 + P_y^2 + P_z^2)/2 \tag{2.115}$$

$$A_1 = P_x^2 - P_y^2 \tag{2.116}$$

$$A_2 = P_y^2 - P_z^2 \tag{2.117}$$

$$A_3 = P_x P_y \tag{2.118}$$

$$A_4 = P_x P_z \tag{2.119}$$

$$A_5 = P_y P_z \tag{2.120}$$

$$L_1 = yP_z - zP_y \tag{2.121}$$

$$L_2 = zP_x - xP_z \tag{2.122}$$

$$L_3 = xP_y - yP_z \; . \tag{2.123}$$

The relations between these constants are summarized in the following Poisson bracket table:

	H	A_1	A_2	A_3	A_4	A_5	L_1	L_2	L_3
H	0	0	0	0	0	0	0	0	0
A_1	0	0	0	0	0	0	$2A_5$	$2A_4$	$-4A_3$
A_2	0	0	0	0	0	0	$-4A_5$	$2A_4$	$-2A_3$
A_3	0	0	0	0	0	0	$-A_4$	A_5	A_1
A_4	0	0	0	0	0	0	A_3	$-A_1-A_2$	$-A_5$
A_5	0	0	0	0	0	0	A_2	$-A_3$	A_4
L_1	0	$-2A_5$	$4A_5$	A_4	$-A_3$	$-A_2$	0	L_3	$-L_2$
L_2	0	$-2A_4$	$-2A_4$	$-A_5$	A_1+A_2	A_3	$-L_3$	0	L_1
L_3	0	$4A_3$	$2A_3$	$-A_1$	A_5	$-A_4$	L_2	$-L_1$	0 .

(2.124)

These define a group which is the semi-direct product [41] of the three dimensional rotation group and the five dimensional abelian group formed by the five symmetric second rank tensors. In general, a free particle in n dimensions will have n^2-1 linearly independent constants of which $n(n-1)/2$ will be the generators of R_n. The others, $n(n+1)/2 - 1$ in number, commute among themselves and are the generators of the abelian group. These constants are second rank tensors formed by taking products of the momenta.

The harmonic oscillator Hamiltonian will be generalized in the next chapter and consideration given to the problem of a plane harmonic oscillator in a uniform magnetic field.

CHAPTER 3

THE HARMONIC OSCILLATOR IN A UNIFORM MAGNETIC FIELD

Introduction

In the previous chapter it has been shown how the accidental degeneracy of the harmonic oscillator may be accounted for by the methods presented in Chapter 1. Although historically, attention has most often been focused upon the accidental degeneracies of the harmonic oscillator [9] and the Kepler problem [4, 5], the problem of the cyclotron motion of a charged particle in a uniform magnetic field possesses certain unique features, among which is that it has two linear constants of the motion. It also has accidental degeneracies of its own [42]. Its Hamiltonian,

$$H = (\bar{P} - \frac{e}{c}\bar{A})^2/2m, \tag{3.1}$$

where, for one choice of gauge

$$A = (-\frac{1}{2}B_0 y, +\frac{1}{2}B_0 x, 0) , \tag{3.2}$$

is quadratic.

As shown in the previous chapter, the constants of the motion of the two dimensional harmonic oscillator K, L, and D arose in a natural fashion as products of linear eigenfunctions of the operator T_H. The eigenvalues of T_H occurred in negative pairs, so that their products, belonging to the sums of the corresponding eigenvalues, were constants

of the motion. Moreover, there was a convenient mapping, the Hopf mapping, which yielded a set of canonical angular coordinates Φ and Ψ, whose conjugate momenta were respectively, H and D.

In the present instance, one finds by expanding Eq. (3.1) and using the gauge in Eq. (3.2) that

$$H = \frac{1}{2m}(P_x^2 + P_y^2) + \frac{e^2B_o^2}{8mc^2}(x^2 + y^2) + \frac{eB_o}{2mc}(yP_x - xP_y). \quad (3.3)$$

Effectively this is the sum of a harmonic oscillator Hamiltonian and a term proportional to the angular momentum.

There is one main difference between the problem of cyclotron motion and the harmonic oscillator. The representation of the harmonic oscillator Hamiltonian, applied to the homogeneous space $H^{(1)}$ has negative pairs of non-zero eigenvalues. All constants of the motion are generated by products of the corresponding eigenfunctions, for which the eigenvalue sum is zero. Thus the lowest order constants of the motion are quadratic, belonging to $H^{(2)}$, and all others are linear combinations of these products. With the cyclotron Hamiltonian there are zero eigenvalues, and therefore there are elements of $H^{(1)}$ which are constants of the motion. Whereas the Poisson bracket of two quadratic constants is quadratic, the Poisson bracket of two linear constants is a scalar. A rather different kind of symmetry group results in the two cases; for the harmonic oscillator one finds the unitary unimodular group, while for cyclotron motion one obtains a group generated by the harmonic oscillator ladder operators.

Classical Zeeman Effect for the Harmonic Oscillator

It is somewhat more instructive to consider a plane isotropic harmonic oscillator in a uniform magnetic field rather than the cyclotron motion exclusively. Its Hamiltonian,

$$H = \frac{1}{2m}(\bar{P} - \frac{e}{c}\bar{A})^2 + \frac{1}{2}m\omega_o^2 r^2$$

$$= \frac{1}{2m}(P_x^2 + P_y^2) + \frac{m}{2}(\omega_o^2 + \omega^2)(x^2 + y^2) + \omega(yP_x - xP_y) \tag{3.4}$$

is also quadratic. In Eq. (3.4) ω is the Larmor frequency $eB/2mc$ and ω_o is the natural frequency of the oscillator. This Hamiltonian reduces, in the limit as $\omega \to 0$ $(B \to 0)$ to that of the plane harmonic oscillator, while as $\omega_o \to 0$ it reduces to that of cyclotron motion.

The matrix representation of this Hamiltonian considered as an operator under Poisson bracket and denoted by T_H is

$$T_H = \begin{bmatrix} 0 & \omega & m(\omega^2 + \omega_o^2) & 0 \\ -\omega & 0 & 0 & m(\omega^2 + \omega_o^2) \\ -\frac{1}{m} & 0 & 0 & \omega \\ 0 & -\frac{1}{m} & -\omega & 0 \end{bmatrix}, \tag{3.5}$$

where the basis is composed of the monomials (x, y, P_x, P_y). The eigenvalues and eigenvectors of T_H are

Eigenvector	Eigenvalue
u	$i[(\omega^2 + \omega_o^2)^{1/2} + \omega]$
u^*	$-i[(\omega^2 + \omega_o^2)^{1/2} + \omega]$
v	$i[(\omega^2 + \omega_o^2)^{1/2} - \omega]$
v^*	$-i[(\omega^2 + \omega_o^2)^{1/2} - \omega]$. (3.6)

By defining

$$r^{\pm} = x \pm iy \tag{3.7}$$

$$P^{\pm} = P_x \pm iP_y \ , \tag{3.8}$$

the eigenvectors u and v can be written as follows:

$$u = [m(\omega^2 + \omega_o^2)]^{1/2} \ r^+ + \frac{i}{\sqrt{m}} P^+ \tag{3.9}$$

$$v = [m(\omega^2 + \omega_o^2)]^{1/2} \ r^- + \frac{i}{\sqrt{m}} P^- \tag{3.10}$$

where u^* and v^* are simply the complex conjugates of u and v respectively. These four eigenfunctions satisfy the following relation

$$\{u^*, u\} = \{v^*, v\} = 4i(\omega^2 + \omega^2)^{1/2} \ . \tag{3.11}$$

The constants of the motion will be products of eigenfunctions, the sum of whose eigenvalues is zero. Hence one establishes the quantities uu^*, vv^*, $u^{*R}v$, and u^Rv^* as constants of the motion where R is a number such that

$$R\lambda_1 = \lambda_2 \ , \tag{3.12}$$

and

$$\lambda_1 = (\omega^2 + \omega_o^2)^{1/2} + \omega \tag{3.13}$$

$$\lambda_2 = (\omega^2 + \omega_o^2)^{1/2} - \omega \ . \tag{3.14}$$

In order to display the symmetry group in a convenient form the following linear combinations are taken as the constants of the motion:

$$H = [\lambda_1 \ uu^* + \lambda_2 \ vv^*]/4\sqrt{\omega^2 + \omega_o^2} \tag{3.15}$$

$$K = [u^R v^* + u^{*R} v]/R^{1/2}(uu^*)^{\frac{R-1}{2}} \tag{3.16}$$

$$L = i[u^R v^* - u^{*R} v]/R^{1/2} \ (uu^*)^{\frac{R-1}{2}} \tag{3.17}$$

$$D = [uu^* - Rvv^*]/R \ , \tag{3.18}$$

where H has been written in accordance with Eq. (1.29). These four quantities satisfy the following relations:

$$T_H(K) = T_H(L) = T_H(D) = 0 \tag{3.19}$$

$$T_K(L) = \alpha D \tag{3.20}$$

$$T_L(D) = \alpha K \tag{3.21}$$

$$T_D(K) = \alpha L, \tag{3.22}$$

where

$$\alpha = 8\sqrt{\omega^2 + \omega_o^2} \ . \tag{3.23}$$

The first of these equations is simply a statement of the fact that K, L, and D are constants of the motion, while the latter three equations show that the symmetry group of the system is SU_2.

If in Eq. (3.12) R is a rational number, then the classical system has bounded closed orbits and a quantum mechanical analog exists for the operators K and L. However if R happens to be irrational, then the orbits are space filling.

Limiting Cases

In taking the limit as $\omega \to 0$, i.e. as the magnetic field is turned off, Eq. (3.4) becomes the Hamiltonian for the two dimensional isotropic harmonic oscillator, the eigenvectors become,

$$u = \sqrt{m}\,\omega_o\, r^+ + \frac{i}{\sqrt{m}} P^+ \tag{3.24}$$

$$v = \sqrt{m}\,\omega_o\, r^- + \frac{i}{\sqrt{m}} P^- \,, \tag{3.25}$$

while the four eigenvalues degenerate into two, namely $\pm\, i\omega_o$, whence $R = 1$. Expressed in terms of the new u and v, the four constants are

$$H = (uu^* + vv^*)/4 \tag{3.26}$$

$$K = uv^* + u^*v \tag{3.27}$$

$$L = i(uv^* - u^*v) \tag{3.28}$$

$$D = uu^* - vv^* \,, \tag{3.29}$$

which are the constants previously obtained for the isotropic oscillator. The commutation rules for K, L, and D also still hold.

In considering the limit as $\omega_o \to 0$ one finds that

$$u = \sqrt{m}\,\omega\, r^{+} + \frac{i}{\sqrt{m}} P^{+} \tag{3.30}$$

and

$$v = \sqrt{m}\,\omega\, r^{-} + \frac{i}{\sqrt{m}} P^{-} . \tag{3.31}$$

The eigenvalues of u and u^* approach $\pm 2i\omega$ in this limit while those belonging to v and v^* both approach zero. Hence, in order to satisfy Eq. (3.12) R must also approach zero. The Hamiltonian in this limit approaches that for pure cyclotron motion as in Eq. (3.3). From the values of the eigenvalues one immediately has two linear constants of the motion, v and v^*, and one quadratic constant uu^*.

The constants can be explicitly derived by considering the commutation rules of K, L, and D in the limit as $\omega_o \to 0$.

Rewriting the commutation relations explicitly gives

$$i\left\{\frac{u^R v^* + u^{*R} v}{R^{1/2}(uu^*)^{\frac{R-1}{2}}},\ \frac{u^R v^* - u^{*R} v}{R^{1/2}(uu^*)^{\frac{R-1}{2}}}\right\} = \alpha\ \frac{uu^* - Rvv^*}{R}, \tag{3.32}$$

$$i\left\{\frac{u^R v^* - u^{*R} v}{R^{1/2}(uu^*)^{\frac{R-1}{2}}},\ \frac{uu^* - Rvv^*}{R}\right\} = \alpha\ \frac{u^R v^* + u^{*R} v}{R^{1/2}(uu^*)^{\frac{R-1}{2}}}, \tag{3.33}$$

and

$$\left\{\frac{uu^* - Rvv^*}{R},\ \frac{u^R v^* + u^{*R} v}{R^{1/2}(uu\ )^{\frac{R-1}{2}}}\right\} = i\alpha\ \frac{u^R v^* - u^{*R} v}{R^{1/2}(uu\ )^{\frac{R-1}{2}}}. \tag{3.34}$$

Multiplying the first of these by R and the latter two by $R^{3/2}$ and then

taking the limit as $\omega_o \to 0$ $(R \to 0)$, results in the equations

$$i\ \{(uu^*)^{1/2}\ (v^*+v),\ (uu^*)^{1/2}\ (v^*-v)\} = 8\ \omega\ uu^* \qquad (3.35)$$

$$i\ \{(uu^*)^{1/2}\ (v^*-v),\ uu^*\} = 0 \qquad (3.36)$$

$$\{uu^*,\ (uu^*)^{1/2}\ (v^* + v)\} = 0\ . \qquad (3.37)$$

Since uu^* is simply twice the cyclotron Hamiltonian, it follows from the last two equations that both the real and imaginary parts of v are constants of the motion. Dividing Eq. (3.35) by uu^* gives

$$\{\frac{v + v^*}{2},\ \frac{v - v^*}{2}\} = 8\omega\ . \qquad (3.38)$$

Expanding Eq. (3.1) results in

$$H = (P_x^2 + P_y^2)/2m + m\omega^2\ (x^2 + y^2)/2 + \omega\ (yP_x - xP_y)\ . \qquad (3.39)$$

Thus the cyclotron Hamiltonian splits into two parts, one being the harmonic oscillator Hamiltonian H_o and the second proportional to the Z-component of the angular momentum L. Both of these terms commute with the total Hamiltonian $H = H_o + L$. Hence another quadratic constant of the motion is

$$D = H_o - L\ . \qquad (3.40)$$

For convenience in notation define

$$S = \frac{\sqrt{m}}{4i}\ (v-v^*) = m\omega y - P_x \qquad (3.41)$$

$$Q = \frac{\sqrt{m}}{4}\ (v-v^*) = m\omega x + P_y\ . \qquad (3.42)$$

With these definitions the following commutation rules hold:

$$\{H, D\} = \{H, S\} = \{H, Q\} = 0 \tag{3.43}$$

$$\{S, Q\} = 2m\omega \tag{3.44}$$

$$\{D, S\} = 2\omega Q \tag{3.45}$$

$$\{D, Q\} = -2\omega S \ . \tag{3.46}$$

These commutation relations coincide with those obtained by Johnson and Lippmann [42]. These authors have discussed the two constants S and Q in considerable detail and show that they are simply related to the location of the center of the circular orbit and to its diameter, which can readily be seen in the following manner. The canonical momentum expressed in terms of the mechanical momentum is

$$\overline{P} = m\overline{v} + \frac{e}{c}\overline{A} \ . \tag{3.47}$$

Substituting for the canonical momenta in Eqs. (3.41) and (3.42) gives

$$S = m(2\omega y - v_x) \tag{3.48}$$

$$Q = m(2\omega x + v_y) \ . \tag{3.49}$$

Evaluating S when $v_x = 0$ and Q when $v_y = 0$ gives the center of the orbit as

$$(x_c, y_c) = (Q/2m\omega, S/2m\omega) . \tag{3.50}$$

These constants also determine the diameter of the orbit. Since the orbit is a circle only one of the constants must be considered. For

example, consider Q. When v_y takes on its maximum positive value, x takes on its minimum value and when v_y takes on its maximum negative value, x is a maximum and hence the diameter d is

$$d = x_{max} - x_{min} = \frac{v_{max}}{\omega} . \tag{3.51}$$

Because of the continuum of points available for the center for a given energy the degeneracy of this problem is infinite.

Canonical Coordinates

As in the case of the plane harmonic oscillator a set of canonical coordinates can be found such that the Hamiltonian becomes a canonical momentum. In fact, two momenta for the problem are:

$$H = \frac{\lambda_1 uu^* + \lambda_2 vv^*}{4(\omega^2 + \omega_o^2)^{1/2}} \tag{3.52}$$

and

$$D = \frac{uu^* - Rvv^*}{R} , \tag{3.53}$$

while the coordinates conjugate to these momenta are

$$Q_1 = \lambda_2 [\ln \frac{u^*}{u} + \lambda_1 \ln \frac{v^*}{v}]/4i \lambda_1 \lambda_2 \tag{3.54}$$

$$Q_2 = \lambda_2 [\ln \frac{u^*}{u} - \lambda_1 \ln \frac{v^*}{v}]/16i \lambda_2 (\omega^2 + \omega_o^2)^{1/2} \tag{3.55}$$

respectively.

A mapping similar to the Hopf mapping may also be performed where

$$u = [\lambda_2^{1/2} \cos \tau \; e^{-i\rho}] \tag{3.56}$$

and

$$v = [\lambda_1^{1/2} \sin \tau \; e^{-i\sigma}] \quad . \tag{3.57}$$

Under this mapping the momenta become

$$H = \lambda_1 \lambda_2 / 4 \; (\omega^2 + \omega_o^2)^{1/2} \tag{3.58}$$

$$D = \lambda_1 \; \cos \theta \tag{3.59}$$

where

$$\theta = 2\tau \tag{3.60}$$

and the coordinates are

$$Q_1 = \Psi / 2\lambda_2 \tag{3.61}$$

and

$$Q_2 = \Phi / 8R(\omega^2 + \omega_o^2)^{1/2}, \tag{3.62}$$

where

$$\Psi = R\rho + \sigma \tag{3.63}$$

and

$$\Phi = R\rho - \sigma \; . \tag{3.64}$$

Performing the mapping on the other two constants of the motion gives

$$K = \lambda_1 \; \sin \theta \cos \Phi \tag{3.65}$$

and

$$L = \lambda_1 \; \sin \theta \sin \Phi . \tag{3.66}$$

There also exists another set of canonical coordinates which were originally defined by Goshen and Lipkin [24]. Written in terms of the cartesian coordinates the momenta are taken to be

$$P_q = [(P_x^2 + P_y^2)/2m + m(\omega^2 + \omega_o^2)(x^2 + y^2)/2]/(\omega^2 + \omega_o^2)^{1/2} \quad (3.67)$$

and

$$P_\theta = xP_y - yP_x \ . \quad (3.68)$$

The corresponding coordinates are

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{[P_x P_y/m + m(\omega^2 + \omega_o^2)\ xy]}{(P_x^2 + P_y^2)/2m + m(\omega^2 + \omega_o^2)(x^2 + y^2)/2} \right) \quad (3.69)$$

and

$$q = \frac{1}{2} \tan^{-1} \left(\frac{(\omega^2 + \omega_o^2)^{1/2}\ (xP_x + yP_y)}{(P_x^2 + P_y^2)/2m - m(\omega^2 + \omega_o^2)(x^2 + y^2)/2} \right) . \quad (3.70)$$

In terms of these variables the Hamiltonian has the particularly simple form

$$H = (\omega^2 + \omega_o^2)^{1/2}\ P_q - \omega P_\theta, \quad (3.71)$$

from which it follows that both P_q and P_θ are constant in time.

Rotating Coordinates

The problem of the plane harmonic oscillator in a uniform magnetic field has a certain uniqueness when viewed from a rotating coordinate system. However, the problem will first be solved in plane polar coordinates. An extremely lucid description and tabulation of these orbits has been given by E.R. Harrison [43].

Assuming the direction of the magnetic field to be in the negative Z direction, the Lagrangian is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{m\omega_o^2}{2} r^2 - m\omega r^2\dot{\theta} \tag{3.72}$$

where the following gauge has been chosen,

$$A_r = A_z = 0 \tag{3.73}$$

and

$$A_\theta = -\frac{1}{2} B_o r \ . \tag{3.74}$$

The momenta conjugate to r and θ are

$$P_r = m\dot{r} \tag{3.75}$$

and

$$P_\theta = mr^2(\dot{\theta} - \omega) . \tag{3.76}$$

Since L is cyclic in θ, P_θ is a constant of the motion. The Hamiltonian is

$$\begin{aligned} H &= P_r\dot{r} + P_\theta\dot{\theta} - L \\ &= (P_r^2 + P_\theta^2/r^2)/2m + m(\omega^2 + \omega_o^2)r^2/2 + \omega P_\theta \ , \end{aligned} \tag{3.77}$$

and the equations of motion are:

$$\dot{P}_r = \frac{P_\theta^2}{mr^3} - m(\omega^2 + \omega_o^2)r \tag{3.78}$$

$$\dot{P}_\theta = 0 \tag{3.79}$$

$$\dot{r} = P_r/m \tag{3.80}$$

$$\dot{\theta} = \frac{P_\theta}{mr^2} + \omega \ . \tag{3.81}$$

In general, the effect of imposing a uniform magnetic field on a system

with a central potential is to add two terms to the Hamiltonian, namely a harmonic oscillator potential which is often neglected for small fields [44], and a term proportional to P_θ, the angular momentum.

The solutions for the orbit equations are:

$$\theta - \theta_o = \frac{a}{2|a|} \sin^{-1} \left\{ \frac{br^2 - 2a^2}{r^2[b^2 - 4a^2W^2]^{1/2}} \right\} + \frac{\omega}{2W} \sin^{-1} \left\{ \frac{2W^2r^2 - b}{[b^2 - 4a^2W^2]^{1/2}} \right\}$$

$$- \frac{a}{2|a|} \sin^{-1} \left\{ \frac{br_o^2 - 2a^2}{r_o^2[b^2 - 4a^2W^2]^{1/2}} \right\} - \frac{\omega}{2W} \sin^{-1} \left\{ \frac{2W^2r_o^2 - b}{[b^2 - 4a^2W^2]^{1/2}} \right\} \tag{3.82}$$

where

$$b = \frac{a^2}{r_o^2} + W^2r_o^2 + \dot{r}_o^2 \tag{3.83}$$

$$a = P_\theta/m \tag{3.84}$$

$$W = (\omega^2 + \omega_o^2)^{1/2} . \tag{3.85}$$

The subscripts on the coordinates and velocities denote initial values. These orbits are plotted in Fig.3.1 through 3.4. In all cases units have been chosen such that $m = q = c = 1$. Fig. 3.1 and 3.2 show the high and low field orbits respectively, for the same set of initial conditions and ω_o. Fig. 3.3 and 3.4 show the orbits for a fixed magnetic field, but for different values of the initial tangential velocity. In general the orbit will be a precessing ellipse for $P_\theta > 0$ and will be a hypotrachoid similar to Fig. 3.1 for $P_\theta < 0$.

Parameters		Initial Conditions	
$\omega = 63/4$		$r_o = 1$	$\dot{r}_o = 0$
$\omega_o = 4$		$\theta_o = 0$	$\dot{\theta}_o = 1$

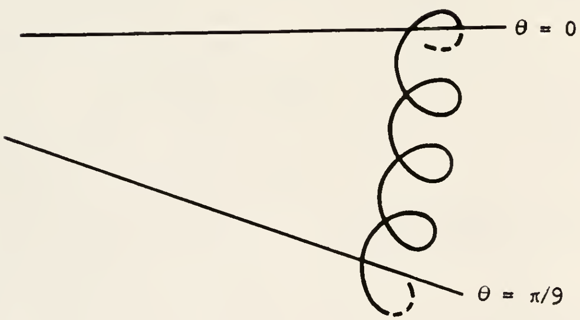


Fig. 3.1 Particle orbit in a strong uniform magnetic field and harmonic oscillator potential.

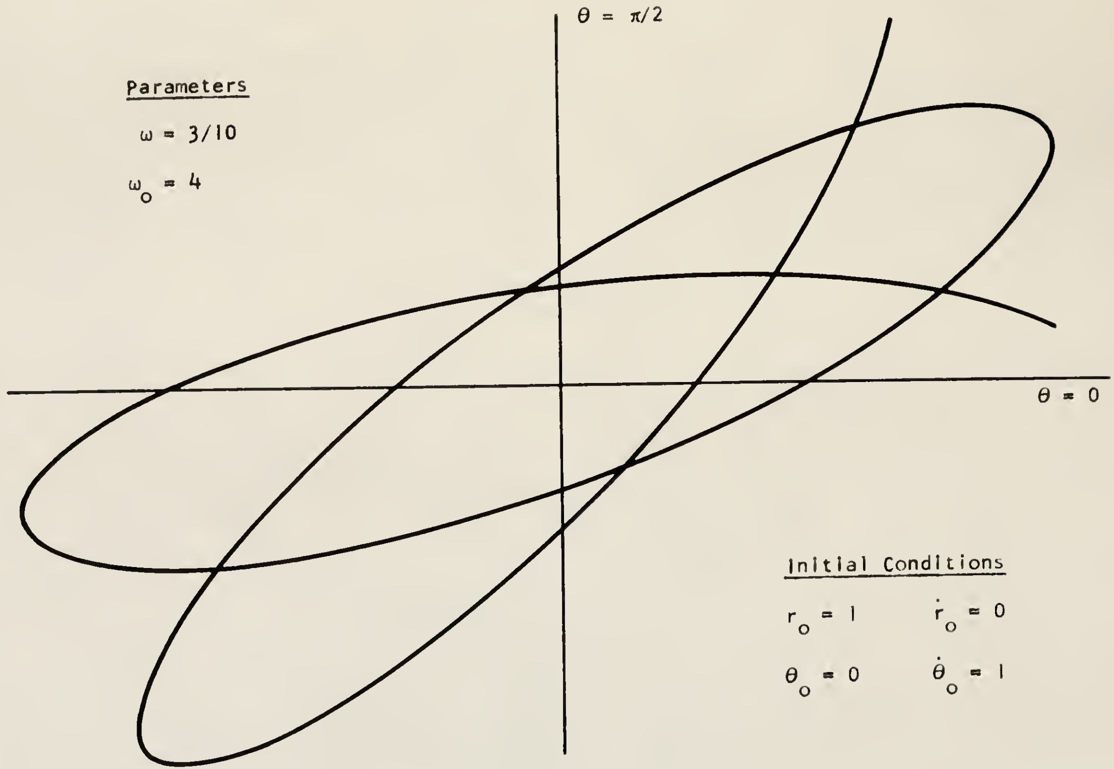
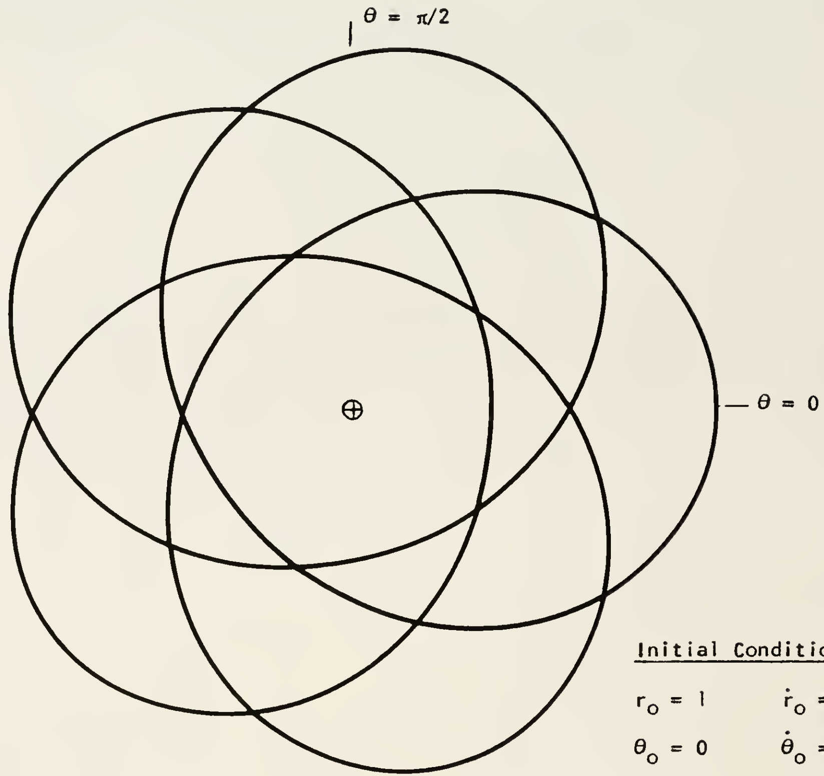


Fig. 3.2 Particle orbit in a weak uniform magnetic field and harmonic oscillator potential.



Parameters

$\omega = 3$

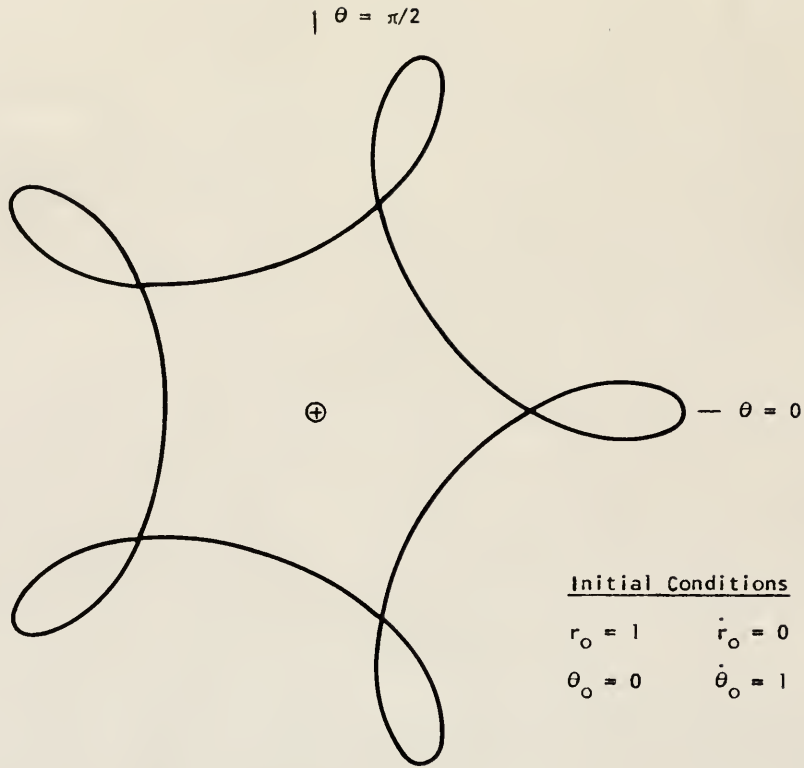
$\omega_o = 4$

Initial Conditions

$r_o = 1$ $\quad$ $\dot{r}_o = 0$

$\theta_o = 0$ $\quad$ $\dot{\theta}_o = 5$

Fig. 3.3 Particle orbit in a fixed uniform magnetic field and harmonic oscillator potential with a large initial tangential velocity.



<u>Parameters</u>

$\omega = 3$

$\omega_o = 4$

<u>Initial Conditions</u>

$r_o = 1$ $\quad \dot{r}_o = 0$

$\theta_o = 0$ $\quad \dot{\theta}_o = 1$

Fig. 3.4 Particle orbit in a fixed uniform magnetic field and harmonic oscillator potential with a small initial tangential velocity.

In a rotating coordinate system defined by

$$\bar{\theta} = \theta - \omega t \tag{3.86}$$

$$\bar{r} = r \ , \tag{3.87}$$

where ω is the Larmor frequency, the Lagrangian is

$$\bar{L} = \frac{m}{2}\left[\dot{\bar{r}}^2 + \bar{r}^2\left(\dot{\bar{\theta}}^2 - \omega^2\right)\right] - \frac{m\omega_o^2}{2}\bar{r}^2 . \tag{3.88}$$

The canonical momenta in the rotating coordinates system are:

$$P_{\bar{r}} = m\dot{\bar{r}} \tag{3.89}$$

$$P_{\bar{\theta}} = m\bar{r}^2\dot{\bar{\theta}} \quad . \tag{3.90}$$

$\bar{L}$ is also cyclic in $\bar{\theta}$ and hence $P_{\bar{\theta}}$ is a constant. The Hamiltonian in the rotating coordinate system is

$$\bar{H} = P_r\dot{\bar{r}} + P_{\bar{\theta}}\dot{\bar{\theta}} - \bar{L}$$

$$= \left(P_{\bar{r}}^2 + P_{\bar{\theta}}^2/\bar{r}^2\right)/2m + m\left(\omega^2 + \omega_o^2\right)\bar{r}^2/2 \tag{3.91}$$

and hence the equations of motion are

$$\dot{P}_{\bar{r}} = \frac{P_{\bar{\theta}}^2}{m\bar{r}^3} - m\left(\omega^2 + \omega_o^2\right)\bar{r} \tag{3.92}$$

$$\dot{P}_{\bar{\theta}} = 0 \tag{3.93}$$

$$\dot{\bar{r}} = P_{\bar{r}}/m \tag{3.94}$$

$$\dot{\bar{\theta}} = \frac{P_{\theta}}{m\bar{r}^2} \quad , \tag{3.95}$$

which are the equations of motion for a plane isotropic harmonic oscillator with force constant $m(\omega^2 + \omega_0^2)$. In general the transformation of a Hamiltonian with a central force from a stationary coordinate system to a rotating coordinate system will subtract a term proportional to P_θ. Generalizing then, it can be stated that a Hamiltonian with a central potential and with a uniform magnetic field present, when viewed from a rotating coordinate system, has a harmonic oscillator potential added, that is

$$\bar{H} = \bar{T} + V(\bar{r}) + \frac{1}{2} m\omega^2 \bar{r}^2 \tag{3.96}$$

where $\bar{T}$ is the kinetic energy in the rotating coordinate system and ω is the rotation frequency of the coordinate system with respect to the fixed system. In the present case since $V(r)$ is a harmonic oscillator potential one simply has a harmonic oscillator with a larger force constant.

The solution for the orbit defined by the Eqs. (3.92) through (3.95) is

$$\bar{\theta} - \bar{\theta}_0 = \frac{a}{2|a|} \sin^{-1} \left(\frac{b\bar{r}^2 - 2a^2}{\bar{r}^2 [b^2 - 4a^2 W^2]^{1/2}} \right) - \frac{a}{2|a|} \sin^{-1} \left(\frac{b\bar{r}_0^2 - 2a^2}{\bar{r}_0^2 [b^2 - 4a^2 W^2]^{1/2}} \right), \tag{3.97}$$

where W is defined in Eq. (3.85) and

$$a = P_{\bar{\theta}}/m \tag{3.98}$$

$$b = a^2/\bar{r}_0^2 + W^2 \bar{r}_0^2 + \dot{\bar{r}}_0^2 . \tag{3.99}$$

Equation (3.97) is the equation for an ellipse, regardless of the

magnitude of ω. The orbits corresponding to Fig. 3.1 and 3.2 in a rotating coordinate system are shown in Fig. 3.5 and 3.6.

Discussion of the Constants of the Motion

Of the four constants of the motion defined in Eqs. (3.15) through (3.18) one is H, the energy of the system. Another constant is the angular momentum P_θ which is a linear combination of H and D

$$P_\theta = \frac{\omega}{\lambda_1 \lambda_2} H - \frac{1 + R}{4(\omega^2 + \omega_o^2)^{1/2}} D. \tag{3.100}$$

Since only three of the constants are independent one would like to discover one other constant, independent of the two above, which has a physical meaning. This can be done by considering the constants of the motion in the rotating coordinate system and then transforming back to the original coordinate system.

In the rotating coordinate system two constants of the motion are aa^* and bb^*, where

$$a = P_{\bar{x}} - i(\omega^2 + \omega_o^2)^{1/2} \bar{x} \tag{3.101}$$

and

$$b = P_{\bar{y}} - i(\omega^2 + \omega_o^2)^{1/2} \bar{y}, \tag{3.102}$$

and where the mass is taken to be unity. By a simple rotation the axis can be oriented so that the semi-major axis of the ellipse lies along $\bar{x}$ and the semi-minor axis lies along $\bar{y}$. In this case then aa^* is proportional to the maximum value which $\bar{x}^2$ attains, because since

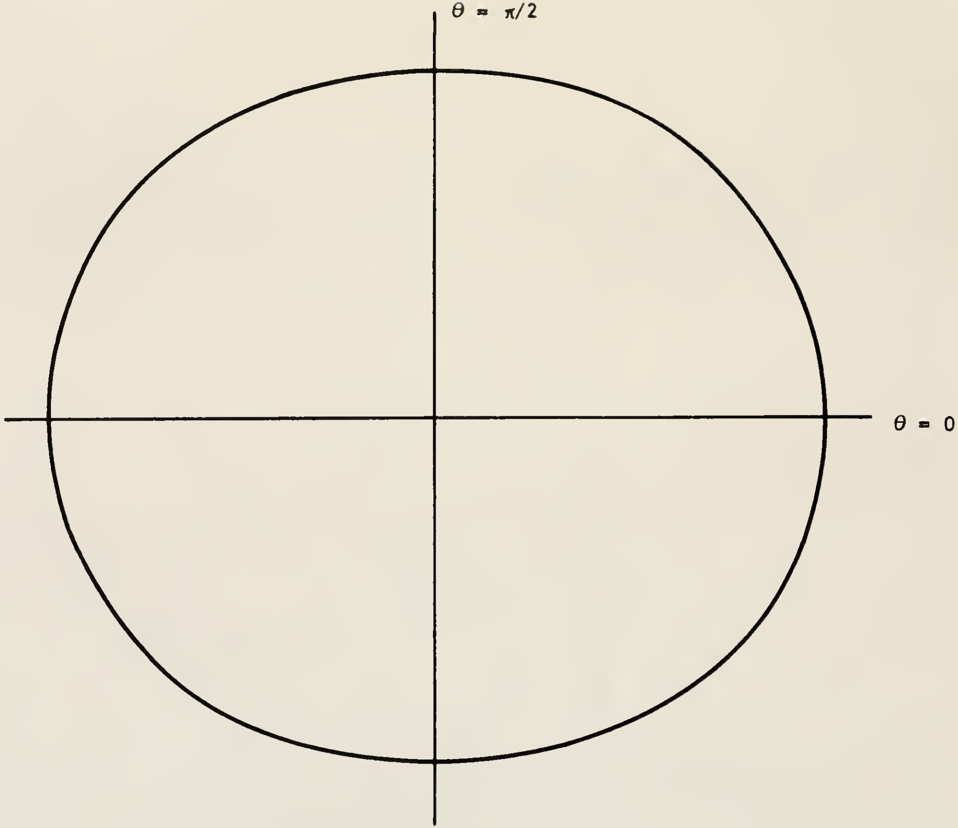


Fig. 3.5 Particle orbit shown in Fig. 3.1 as viewed from a rotating coordinate system.

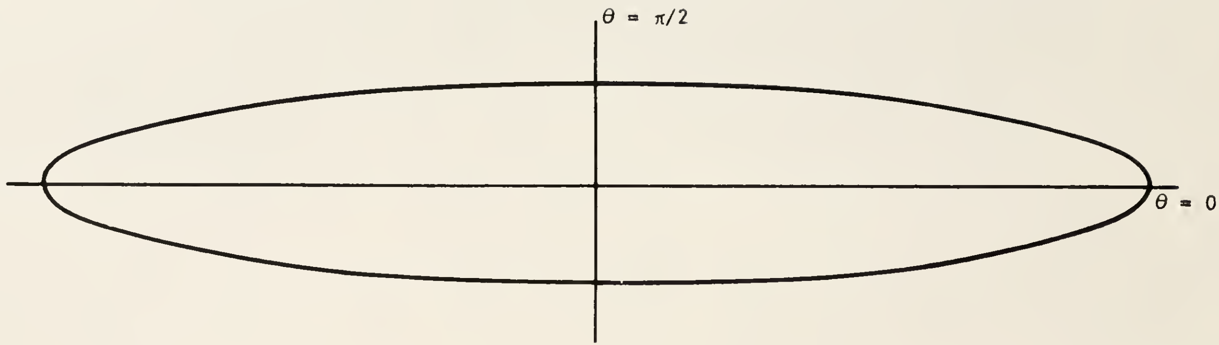


Fig. 3.6 Particle orbit shown in Fig. 3.2 as viewed from a rotating coordinate system.

$$aa^* = P_{\bar{x}}^2 + (\omega^2 + \omega_o^2)\ \bar{x}^2 \qquad (3.103)$$

is a constant, it can be evaluated when $P_{\bar{x}} = 0$. Similarly bb^* is proportional to $\bar{y}^2_{max}$. Alternatively, one can say that aa^* and bb^* give the boundaries of the orbit i.e. aa^* gives the maximum radial distance the particle can attain and bb^* is the minimum radial distance.

These two radii are also constants of the motion in the stationary system since the transformation only involved the angle. However, it can explicitly be shown that these are constants.

By transforming from the rotating coordinate system to the stationary one aa^* can be written in terms of the eigenvectors and their conjugates as

$$aa^* = (1/4)(uu^* + vv^* + uv^*\ e^{2i\omega t} + u^*v\ e^{-2i\omega t})\ . \qquad (3.104)$$

In the stationary system the time rate of change is

$$\frac{d}{dt}(aa^*) = \{aa^*, H\} + \frac{\partial}{\partial t}(aa^*)$$

$$= (1/4)[-2i\omega uv^* e^{2i\omega t} + 2i\omega u^* v e^{-2i\omega t}$$

$$+ 2i\omega uv^* e^{2i\omega t} - 2i\omega u^* v e^{-2i\omega t}] = 0, \qquad (3.105)$$

and hence aa^* is a constant of the motion in the stationary system also. A similar proof also shows bb^* to be a constant. However aa^* and bb^* are not independent, but are related to one another through H and D.

Even though a good physical interpretation cannot be given to the constants K, L, and D their effect on the orbit under Poisson bracket can be calculated i.e., the infinitesimal change each produces.

Before calculating the effects of the constants on the orbit, it is desirable to write the orbit equation in terms of the constants of the motion and the time. This can be done by defining a complex vector

$$\bar{r} = x + iy \qquad (3.106)$$

which can be written in terms of the eigenvectors by inverting Eqs. (3.9) and (3.10)

$$\bar{r} = (u + v^*)/2c \ , \qquad (3.107)$$

where

$$c = [m(\omega^2 + \omega_o^2)]^{1/2}. \qquad (3.108)$$

Employing the polar forms of both the eigenvectors in Eqs. (3.56) and (3.57) and of the constants in Eqs. (3.59), (3.65) and (3.66), u and v^* may be expressed in terms of the constants as

$$u = (R/2)^{1/2} [(K^2 + L^2 + D^2)^{1/2} + D]^{1/2} e^{-i\rho} \qquad (3.109)$$

$$v^* = (R/2)^{1/2} [(K^2 + L^2 + D^2)^{1/2} - D]^{1/2} e^{i\sigma} \ . \qquad (3.110)$$

However from Eqs. (3.63) and (3.64)

$$\rho = (1/2R)(\Psi + \Phi) \qquad (3.111)$$

and

$$\sigma = (1/2)(\Psi - \Phi) \ . \qquad (3.112)$$

Hence

$$u = (R/2)^{1/2} [(K^2 + L^2 + D^2)^{1/2} + D]^{1/2} e^{-i\Phi/2R} e^{-i\Psi/2R} \qquad (3.113)$$

and

$$v^* = (R/2)^{1/2} [(K^2 + L^2 + D^2)^{1/2} - D]^{1/2} e^{-i\Phi/2} e^{i\Psi/2} \ . \qquad (3.114)$$

From Eqs. (3.65) and (3.66) it is apparent that

$$\phi = \tan^{-1}\frac{L}{K} . \tag{3.115}$$

The time dependence is in Ψ since $\Psi/2\lambda_2$ in Eq. (3.61) is the coordinate canonical to H and hence from

$$d/dt\ (\Psi/2\lambda_2) = \{\Psi/2\lambda_2,\ H\} = 1 \tag{3.116}$$

it follows that

$$\Psi = 2\lambda_2(t - t_o) \tag{3.117}$$

or

$$\Psi = 2R\lambda_1(t - t_o) . \tag{3.118}$$

Substituting Eqs. (3.115), (3.117), and (3.118) into Eqs. (3.113) and (3.114) and then substituting these into Eq. (3.107) gives

$$\bar{r} = \frac{1}{2c}\left(\frac{R}{2}\right)^{1/2}\{[(K^2 + L^2 + D^2)^{1/2}+D]^{1/2} e^{(-i/2R)\tan^{-1}(L/K)}\ e^{-i\lambda_1(t-t_o)}$$

$$+ [(K^2 + L^2 + D^2)^{1/2}-D]^{1/2} e^{(-i/2)\tan^{-1}(L/K)}\ e^{i\lambda_2(t-t_o)}\}$$

$$= \bar{r}_- + \bar{r}_+ , \tag{3.119}$$

where

$$\bar{r}_- = \frac{1}{2c}\left(\frac{R}{2}\right)^{1/2}[(K^2 + L^2 + D^2)^{1/2}+D]^{1/2} e^{(-i/2R)\tan^{-1}(L/K)}\ e^{-i\lambda_1(t-t_o)} \tag{3.120}$$

and

$$\bar{r}_+ = \frac{1}{2c}\left(\frac{R}{2}\right)^{1/2}[(K^2 + L^2 + D^2)^{1/2}-D]^{1/2} e^{(-i/2)\tan^{-1}(L/K)}\ e^{i\lambda_2(t-t_o)} . \tag{3.121}$$

Equation (3.119) expresses the orbit equation as the sum of two contra rotating vectors, whose frequencies of rotation are λ_1 and λ_2, and

whose amplitudes are functions of the constants of the motion.

The Poisson bracket of the constants K, L, and D with $\bar{r}$ are:

$$\{K, \bar{r}\} = \frac{\alpha}{2}\left[\frac{L}{[(K^2 + L^2 + D^2)^{1/2} - D]} - \frac{iKD}{K^2 + L^2}\right]\bar{r}_+$$

$$- \frac{\alpha}{2}\left[\frac{L}{[(K^2 + L^2 + D^2)^{1/2} + D]} + \frac{iKD}{R(K^2 + L^2)}\right]\bar{r}_- \qquad (3.122)$$

$$\{L, \bar{r}\} = -\frac{\alpha}{2}\left[\frac{K}{[(K^2 + L^2 + D^2)^{1/2} - D]} + \frac{iLD}{K^2 + L^2}\right]\bar{r}_+$$

$$+ \frac{\alpha}{2}\left[\frac{K}{[(K^2 + L^2 + D^2)^{1/2} + D]} - \frac{iLD}{R(K^2 + L^2)}\right]\bar{r}_- \qquad (3.123)$$

$$\{D, \bar{r}\} = \frac{i\alpha}{2R}\bar{r}_- + \frac{i\alpha}{2}\bar{r}_+ \qquad (3.124)$$

where α is a constant defined in Eq. (3.23).

The infinitesimal change induced by the constants is that each changes both the amplitude and phase of r_+ and r_-, while preserving the sum of the squares of their sum and difference. If the orbit were an ellipse, this last statement would be equivalent to stating that the sum of the squares of the semi axes of the ellipse is a constant. The proof of the statement goes as follows:

Before the infinitesimal change the quantity is

$$|r_+ + r_-|^2 + |r_+ - r_-|^2 = 2(|r_+|^2 + |r_-|^2) \qquad (3.125)$$

and after the change it is

$$|r_+(1 + \epsilon) + r_-(1 + \delta)|^2 + |r_+(1 + \epsilon) - r_-(1 - \delta)|^2$$

$$= 2[|r_+|^2 + |r_-|^2 + (\epsilon + \epsilon^*)\ |r_+|^2 + (\delta + \delta^*)\ |r_-|^2] \quad (3.126)$$

where second order terms have been ignored. The difference between the two terms is

$$2[(\epsilon + \epsilon^*)\ |r_+|^2 + (\delta + \delta^*)\ |r_-|^2]\ ,$$

which is zero in all three cases. For example, if the infinitesimal change is induced by K, one has for the difference

$$2[(\epsilon + \epsilon^*)|r_+|^2 + (\delta + \delta^*)|r_-|^2]$$

$$= \frac{2\alpha}{2}\left[\frac{2L\ |r_+|^2}{[(K^2 + L^2 + D^2)^{1/2} - D]} - \frac{2L\ |r_-|^2}{[(K^2 + L^2 + D^2)^{1/2} + D]}\right]$$

$$= \frac{2\alpha R}{8c}\left[\frac{L[(K^2 + L^2 + D^2)^{1/2} - D]}{[(K^2 + L^2 + D^2)^{1/2} - D]} - \frac{L[(K^2 + L^2 + D^2)^{1/2} + D]}{[(K^2 + L^2 + D^2)^{1/2} + D]}\right]$$

$$= 0. \quad (3.127)$$

Gauge Transformations in Uniform Magnetic Fields

In uniform magnetic fields and in the absence of other external potentials the following fact concerning a charged particle in this field is true: The change in momentum in going from one point to another is independent of the path taken between the points. The

proof is as follows: Let $\bar{P}$ denote the canonical momentum, π the mechanical momentum, $\bar{B}$ the uniform field, and $\bar{A}$ the vector potential such that

$$\bar{B} = \nabla \times \bar{A} \tag{3.128}$$

and since $\bar{B}$ is uniform

$$\bar{A} = \frac{1}{2} \bar{B} \times \bar{r} \quad . \tag{3.129}$$

Since

$$\frac{d\bar{\pi}}{dt} = \frac{e}{c} \nabla \times \bar{B} \ . \tag{3.130}$$

and

$$\bar{v} = \frac{d\bar{r}}{dt} \tag{3.131}$$

one has

$$\begin{aligned} \int_{\bar{r}_1}^{\bar{r}_2} d\bar{\pi} &= \frac{e}{c} \int_{\bar{r}_1}^{\bar{r}_2} \left(\frac{d\bar{r}}{dt} \times \bar{B}\right) dt \\ &= \frac{e}{c} \int_{\bar{r}_1}^{\bar{r}_2} d\bar{r} \times \bar{B} \\ &= - \frac{2e}{c} \int_{\bar{r}_1}^{\bar{r}_2} dA \ . \end{aligned} \tag{3.132}$$

Hence

$$\Delta\bar{\pi} = - \frac{2e}{c} \Delta\bar{A} \tag{3.133}$$

and

$$\Delta\bar{P} = - \frac{e}{c} \Delta\bar{A} \ . \tag{3.134}$$

Equation (3.134) states that the difference in momenta between any two points is proportional to the difference in the vector potential evaluated at these points. From this it is inferred that the choice of a certain gauge is equivalent to picking the zero of momentum.

One can also see why a translation of coordinates must be accompanied by a gauge transformation. When the translation occurs the zero of momentum changes and hence this change must be subtracted from $\overline{A}$ to give the the same zero of momentum. However, the change in $\overline{A}$ is simply a gauge transformation.

In order to show that a gauge transformation can induce a translation, first consider the case of a free particle in a uniform field with the following gauge

$$A = \frac{B_o}{2}(-y, x, 0) \ . \tag{3.135}$$

In units where m and $\omega = eB_o/2mc$ are one, the constants of the motion are

$$P_x - y = \alpha \tag{3.136}$$

and

$$P_y + x = \beta \ . \tag{3.137}$$

Using the fact that

$$\overline{P} = \overline{v} + \frac{e}{c}\overline{A} \ , \tag{3.138}$$

the location of the center of the orbit is found to be

$$(x_c, y_c) = (\frac{\beta}{2}, -\frac{\alpha}{2}) \ . \tag{3.139}$$

However with the gauge

$$A = \frac{B_o}{2}(-(\lambda + y), \mu + x, 0), \tag{3.140}$$

even though the two constants are the same, one finds the center of the orbit to be

$$(x_c, y_c) = (\frac{\beta - \mu}{2} \quad \frac{\alpha - \lambda}{2}) \ . \tag{3.141}$$

Hence the center of the orbit has been translated $-\mu/2$ units in the x direction and $\lambda/2$ units in y.

The center of the cyclotron orbit may also be rotated, however this is not accomplished by a simple gauge transformation. Instead one must rotate the vector $\bar{A}$ and also rotate the coordinate system. For the vector potential defined in Eq. (3.135) the center of the orbit is given in Eq. (3.139). A rotation of the vector field defined by $\bar{A}$ gives

$$\bar{A}_R = \frac{B_o}{2} \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} -y \\ x \\ 0 \end{vmatrix}$$

$$= \frac{B_o}{2} (-y \cos\theta + x \sin\theta, \; y \sin\theta + x \cos\theta, 0). \quad (3.142)$$

The rotation of the coordinates defined by

$$x = x' \cos\theta - y' \sin\theta \quad (3.143)$$

and

$$y = x' \sin\theta + y' \cos\theta \quad (3.144)$$

transforms $\bar{A}_R$ into $\bar{A}'$ where

$$\bar{A}' = \frac{B_o}{2} (-y', x', 0) . \quad (3.145)$$

Hence, the center of the orbit is

$$(x_c', y_c') = (\frac{\beta'}{2}, -\frac{\alpha'}{2}) \quad (3.146)$$

and

$$\frac{\beta}{2} = \frac{\beta'}{2} \cos\theta + \frac{\alpha'}{2} \sin\theta \quad (3.147)$$

$$\frac{\alpha}{2} = -\frac{\beta'}{2}\sin\theta + \frac{\alpha'}{2}\cos\theta, \qquad (3.148)$$

which shows that the center of the orbit has been rotated with respect to the original coordinate system, while the diameter of the orbit has remained unchanged.

The vector field can also be rotated without rotating the coordinates and still preserve the curl if at the same time the vector is dilated by a factor of sec θ. The new vector potential is

$$\bar{A} = \frac{B_0}{2}\sec\theta\,(-y\cos\theta + x\sin\theta,\ y\sin\theta + x\cos\theta,\ 0). \qquad (3.149)$$

The transformation leaves the center of the orbit invariant and in fact, is equivalent to a gauge transformation where the vector $\bar{A}_D$

$$\bar{A}_D = \frac{B_0}{2}\tan\theta(x,\ y,\ 0), \qquad (3.150)$$

whose curl is zero, has been added to the original vector potential $\bar{A}$.

CHAPTER 4

THE KEPLER PROBLEM

The Two Dimensional Case in Polar Coordinates

The Hamiltonian for the two dimensional Kepler problem in polar coordinates is

$$H = (1/2)(P_r^2 + P_\phi^2/r^2) - 1/r \tag{4.1}$$

where the mass and force constant have been chosen equal to unity. The corresponding Hamilton-Jacobi equation is

$$\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 - 2\left(W + \frac{1}{r}\right) = 0. \tag{4.2}$$

where W is the total energy and

$$S(r, \phi) = S_r(r) + S_\phi(\phi) \ . \tag{4.3}$$

Hence

$$P_r = \frac{\partial S}{\partial r} \tag{4.4}$$

and

$$P_\phi = \frac{\partial S}{\partial \phi} \tag{4.5}$$

Since the partial differential equation (4.2) has no ϕ dependence $\partial S/\partial\phi = P_\phi$ is a constant and is, in fact, the angular momentum of the system. Solving Eq. (4.2) for $\partial S/\partial r = P_r$ one finds

$$\frac{\partial S}{\partial r} = P_r = \left[2W + \frac{2}{r} - \frac{P_\phi^2}{r^2}\right]^{1/2} \ . \tag{4.6}$$

The action variables J_r and J_ϕ are given by the integrals

$$J_r = \oint P_r dr \tag{4.7}$$

and

$$J_\phi = \oint P_\phi d\phi \ . \tag{4.8}$$

Equation (4.8) can be integrated immediately since the integral over one cycle is simply an integral from 0 to 2π, hence

$$J_\phi = 2\pi P_\phi \ . \tag{4.9}$$

The integral in Eq. (4.7) can be evaluated by a contour integration as described in Born [35] and the result is

$$J_r = \frac{2\pi}{[-2W]^{1/2}} - J_\phi \ . \tag{4.10}$$

The angle variables w_r and w_ϕ which are canonical coordinates and conjugate to J_r and J_ϕ are given by

$$w_r = \frac{\partial S}{\partial J_r} = \int \frac{\partial P_r}{\partial J_r} dr \tag{4.11}$$

and

$$w_r = \frac{\partial S}{\partial J_\phi} = \int \frac{\partial P_r}{\partial J_\phi} dr + \int \frac{\partial P_\phi}{\partial J_\phi} d\phi \ . \tag{4.12}$$

Upon performing the integrations

$$w_r = (u - \epsilon \sin u)/2\pi \tag{4.13}$$

and

$$w_\phi = [u - \epsilon \sin u - \tan^{-1}(y/x) + \phi]/2\pi, \tag{4.14}$$

where the elliptic anomaly u is defined through

$$r = a(1 - \epsilon \cos u), \tag{4.15}$$

a being the semi-major axis and ϵ being the eccentricity of the ellipse,

$$a = - W/2 \tag{4.16}$$

$$\epsilon^2 = 1 + 2WP_\phi^2 \quad . \tag{4.17}$$

By choosing ϕ_o to be the azimuth of the perihelion,

$$x = r \cos (\phi - \phi_o) \tag{4.18}$$

$$y = r \sin (\phi - \phi_o) \quad , \tag{4.19}$$

and Eq. (4.14) reduces to

$$w_\phi = (u - \epsilon \sin u + \phi_o)/2\pi \quad . \tag{4.20}$$

Solving Eq. (4.10) for the energy in terms of J_r and J_ϕ

$$W = \frac{2\pi^2}{(J_r + J_\phi)^2} \tag{4.21}$$

and hence the frequencies ν_r and ν_ϕ are degenerate i.e.,

$$\nu_r = \nu_\phi = \frac{\partial W}{\partial J_r} = \frac{\partial W}{\partial J_\phi} \quad . \tag{4.22}$$

Thus the quantities aa^*, bb^*, ab^*, and a^*b are constants of the motion where

$$a = (\frac{J_r}{2\pi})^{1/2} \; e^{2\pi i w_r} \tag{4.23}$$

and $$b = \left(\frac{J_\phi}{2\pi}\right)^{1/2} e^{2\pi i w_\phi} \quad . \qquad (4.24)$$

These quantities satisfy the following Poisson bracket relations

$$\{a, a^*\} = \{b, b^*\} = i \qquad (4.25)$$

where one uses the fact that the J's and w's form a set of canonical coordinates and momenta.

Taking the real and imaginary parts of these constants and the sum and difference of the real ones gives the following four constants of the motion,

$$H = aa^* + bb^* \qquad (4.26)$$

$$D = aa^* - bb^* \qquad (4.27)$$

$$K = ab^* + a^*b \qquad (4.28)$$

$$L = i(ab^* + a^*b). \qquad (4.29)$$

Of these constants H commutes with the other three and is a function of the Hamiltonian, namely, it is inversely proportional to the square root of the energy. The three constants K, D, and L satisfy the commutation relations of the generators of the group SU_2 or R_3.

$$\{K, D\} = 2L \qquad (4.30)$$

$$\{D, L\} = 2K \qquad (4.31)$$

$$\{L, K\} = 2D \quad . \qquad (4.32)$$

Although they have no simple interpretation as constants of the motion, one can make a mapping of these constants which preserves the commutation rules. The new constants can be interpreted as the angular

momentum and the two components of the Runge vector [11]. The mapping is

$$K \rightarrow K' = K\left(2 + \frac{H+D}{H-D}\right)^{1/2} \qquad (4.33)$$

$$L \rightarrow L' = L\left(2 + \frac{H+D}{H-D}\right)^{1/2} \qquad (4.34)$$

$$D \rightarrow D' = H - D\ . \qquad (4.35)$$

These constants satisfy the rules

$$\{K', D'\} = -2L' \qquad (4.36)$$

$$\{D', L'\} = -2K' \qquad (4.37)$$

$$\{L', K'\} = -2D'\ . \qquad (4.38)$$

Otherwise they are a direct consequence of the commutation rules for K, L, D, so that this mapping is valid for any SU_2 group where use must be made of the identity

$$H^2 = K^2 + L^2 + D^2\ . \qquad (4.39)$$

K' is essentially the x component of the vector $\overline{P}$ where

$$\overline{P} = \overline{R}\sqrt{-2E} = (-\epsilon\sqrt{-2E}) \begin{vmatrix} \cos\Phi_o \\ \sin\Phi_o \end{vmatrix} \qquad (4.40)$$

and $\overline{R}$ is the Runge vector, while L' is the y component and D' is the angular momentum.

The Kepler Problem in Three Dimensions

Using the same units as in the previous case, the Hamiltonian for the Kepler problem in three dimensions is

$$H = \frac{1}{2}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2 \sin^2\theta}\right) - \frac{1}{r} \; . \tag{4.41}$$

Setting up the Hamilton-Jacobi equation and separating gives

$$P_\phi = \frac{\partial S}{\partial \phi} = \alpha_\phi = \text{constant} \tag{4.42}$$

$$P_\theta = \frac{\partial S}{\partial \theta} = \left(\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}\right)^{1/2} \tag{4.43}$$

and

$$P_r = \frac{\partial S}{\partial r} = \left(2W + \frac{2}{r} - \frac{\alpha_\theta^2}{r^2}\right)^{1/2}, \tag{4.44}$$

where α_θ and α_ϕ are separation constants, α_ϕ being the magnitude of the Z component of the angular momentum and α_θ the magnitude of the total angular momentum. Since the potential is central the orbit is planar and the normal to the plane is necessarily parallel to the angular momentum. Denoting by B the dihedral angle which the orbital plane makes with the x-y plane one can deduce that

$$\cos B = \frac{\alpha_\phi}{\alpha_\theta} \; . \tag{4.45}$$

Solving for the action variables gives

$$J_\phi = 2\pi\alpha_\phi = 2\pi P_\phi \tag{4.46}$$

$$J_\theta = 2(\alpha_\theta - \alpha_\phi) \tag{4.47}$$

$$J_r = \frac{2}{\sqrt{-2W}} - J_\theta - J_\phi \tag{4.48}$$

and from Eq. (4.48) it follows that

$$W = \frac{-2\pi^2}{(J_r + J_\theta + J_\phi)^2} \quad . \tag{4.49}$$

In the plane of the orbit rectangular coordinates μ and ν can be introduced, where the μ axis is the major axis of the ellipse and the origin coincides with the center of force and the origin of the polar system. In this system

$$\mu = r \cos (\Psi - \Psi_o) \tag{4.50}$$

$$\nu = r \sin (\Psi - \Psi_o) , \tag{4.51}$$

where Ψ is an angle measured from the μ axis to the particle and Ψ_o is the angle to the perihelion.

The angle variables are

$$w_r = \int \frac{\partial p_r}{\partial J_r} dr = (u - \epsilon \sin u)/2\pi \tag{4.52}$$

$$w_\theta = \int \frac{\partial p_r}{\partial J_\theta} dr + \int \frac{\partial p_\theta}{\partial J_r} d\theta$$

$$= (u - \epsilon \sin u - \tan^{-1} \frac{\nu}{\mu} + \sin^{-1} \frac{\cos \theta}{\sin \beta})/2\pi \tag{4.53}$$

$$w_\Phi = \int \frac{\partial P_r}{\partial J_\Phi}\,dr + \int \frac{\partial P_\theta}{\partial J_\Phi}\,d\theta + \int \frac{\partial P_\Phi}{\partial J_\Phi}\,d\Phi$$

$$= \left(u - \epsilon \sin u - \tan^{-1}\frac{\nu}{\mu} + \sin^{-1}\frac{\cos\theta}{\sin B} - \sin^{-1}\frac{\cot\theta}{\tan B} + \Phi\right)/2\pi \qquad (4.54)$$

where u is defined as in Eq. (4.15) and ϵ as in Eq. (4.17). Using Eqs. (4.50) and (4.51) it follows that

$$\tan^{-1}\frac{\nu}{\mu} = \Psi - \Psi_o\ . \qquad (4.55)$$

It follows from Fig. 4.1 that $\sin^{-1}\frac{\cot\theta}{\tan B}$ is the projection of the angle Ψ onto the plane in which Φ is measured i.e., the x-y plane, which shall be denoted $\frown$. It also follows that $\sin^{-1}\frac{\cos\theta}{\sin B}$ is the angle Ψ, where the line of nodes has been chosen to coincide with the major axis of the ellipse. Hence, the line of nodes makes an angle $(\Phi - \frown)$ with the x-axis. Rewriting Eqs. (4.53) and (4.54) results in the following set of equations,

$$w_\theta = (u - \epsilon \sin u + \Psi_o)/2\pi \qquad (4.56)$$

$$w_\Phi = (u - \epsilon \sin u + \Psi_o + \Phi - \frown)/2\pi\ . \qquad (4.57)$$

By defining

$$a = \left(\frac{J_r}{2\pi}\right)^{1/2} e^{2\pi i w_r} \qquad (4.58)$$

$$b = \left(\frac{J_\theta}{2\pi}\right)^{1/2} e^{2\pi i w_\theta} \qquad (4.59)$$

$$c = \left(\frac{J_\Phi}{2\pi}\right)^{1/2} e^{2\pi i w_\Phi} \qquad (4.60)$$

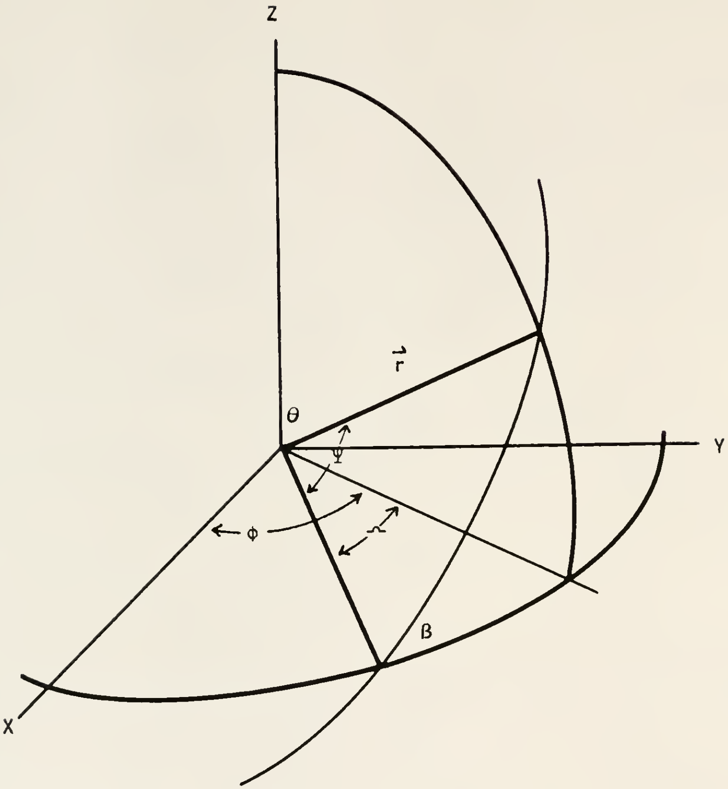


Fig. 4.1 Three dimensional coordinate system for the Kepler problem.

one finds the following nine constants of the motion: aa^*, bb^*, cc^*, ab^*, a^*b, bc^*, b^*c, ac^*, and a^*c. These constants are the generators of the group SU_3. The constant angles β, Ψ_o and $\Phi - \Omega$ and also the sum $\Phi - \Omega + \Psi_o$ can be expressed in terms of the constants,

$$\sin \Psi_o = \frac{i(ab^* - a^*b)}{2(aa^*bb^*)^{1/2}} \tag{4.61}$$

$$\cos \Psi_o = \frac{ab^* + a^*b}{2(aa^*bb^*)^{1/2}} \tag{4.62}$$

$$\sin(\Phi - \Omega) = \frac{i(bc^* - b^*c)}{2(bb^*cc^*)^{1/2}} \tag{4.63}$$

$$\cos(\Phi - \Omega) = \frac{(bc^* + b^*c)}{2(bb^*cc^*)^{1/2}} \tag{4.64}$$

$$\sin(\Phi - \Omega + \Psi_o) = \frac{i(ac^* - a^*c)}{2(aa^*cc^*)^{1/2}} \tag{4.65}$$

$$\cos(\Phi - \Omega + \Psi_o) = \frac{ac^* + a^*c}{2(aa^*cc^*)^{1/2}} \tag{4.66}$$

and

$$\cos \beta = \frac{cc^*}{bb^* + cc^*} . \tag{4.67}$$

Generally, the rotation group in four dimensions R_4, is taken to be the symmetry group of the Kepler problem. Its generators are the three components of the angular momentum vector $\overline{L}$ and the three components of the normalized Runge vector $\overline{P}$ which is defined in terms of $\overline{R}$ in Eq. (4.40). The rectangular components of $\overline{P}$ are

$$\bar{P} = \frac{\epsilon}{\sqrt{-2W}} \begin{vmatrix} \cos\Psi_o \cos(\phi - \Omega) - \cos\beta \sin\Psi_o \sin(\phi - \Omega) \\ -\cos\Psi_o \sin(\phi - \Omega) - \cos\beta \sin\Psi_o \cos(\phi - \Omega) \\ \sin\beta \sin\Psi_o \end{vmatrix} . \tag{4.68}$$

For $\bar{L}$ one has

$$\bar{L} = \left(\frac{J_\theta + J_\phi}{2\pi}\right) \begin{vmatrix} \sin\beta \sin(\phi - \Omega) \\ \sin\beta \cos(\phi - \Omega) \\ \cos\beta \end{vmatrix} . \tag{4.69}$$

Using Eqs. (4.61) through (4.67) one can express the generators of R_4 in terms of the generators of SU_3 as follows:

$$P_x = \frac{(aa^* + 2bb^* + 2cc^*)^{1/2}}{4bb^*(cc^*)^{1/2}(bb^* + cc^*)} [(ab^* + a^*b)(bc^* + b^*c)(bb^* + cc^*) + (ab^* - a^*b)(bc^* - b^*c)\ cc^*] \tag{4.70}$$

$$P_y = \frac{-i(aa^* + 2bb^* + 2cc^*)^{1/2}}{4bb^*(cc^*)^{1/2}(bb^* + cc^*)} [(ab^* + a^*b)(bc^* - b^*c)(bb^* + cc^*) + (ab^* - a^*b)(bc^* + b^*c)\ cc^*] \tag{4.71}$$

$$P_z = \frac{i(aa^* + 2bb^* + 2cc^*)^{1/2}}{2(bb^* + cc^*)} (bb^* + 2cc^*)^{1/2}(ab^* - a^*b) \tag{4.72}$$

$$L_x = \frac{i(bc^* - b^*c)(bb^* + 2cc^*)^{1/2}}{2(cc^*)^{1/2}} \tag{4.73}$$

$$L_y = \frac{(bc^* + b^*c)(bb^* + 2cc^*)^{1/2}}{2(cc^*)^{1/2}} \tag{4.74}$$

and
$$L_z = cc^* \; . \tag{4.75}$$

These six quantities satisfy the relations

$$\{L_i, P_j\} = - \epsilon_{ijk} P_k \tag{4.76}$$

$$\{L_i, L_j\} = - \epsilon_{ijk} L_k \tag{4.77}$$

$$\{P_i, P_j\} = - \epsilon_{ijk} L_k \tag{4.78}$$

which define R_4, once again directly from the commutation rules for SU_3.

The process can be inverted and the generators of SU_3 can be expressed in terms of the components of $\overline{P}$ and $\overline{L}$. These are

$$aa^* = \frac{1}{\sqrt{-2W}} - (L_x^2 + L_y^2)^{1/2} - L_z \tag{4.79}$$

$$bb^* = (L_x^2 + L_y^2)^{1/2} \tag{4.80}$$

$$cc^* = L_z \tag{4.81}$$

$$ab^* = \left[\frac{1}{\sqrt{-2W}} - (L_x^2 + L_y^2)^{1/2} - L_z\right]^{1/2} (L_x^2 + L_y^2)^{1/2} (\cos \Psi_o - i \sin \Psi_o) \tag{4.82}$$

$$bc^* = [L_z (L_x^2 + L_y^2)^{1/2}]^{1/2} \; (\cos (\phi - \Omega) - i \sin (\phi - \Omega)) \tag{4.83}$$

$$ac^* = \left[\frac{1}{\sqrt{-2W}} - (L_x^2 + L_y^2)^{1/2} - L_z\right]^{1/2} L_z^{1/2} [\cos(\phi - \Omega + \Psi_o) - i \sin(\phi - \Omega + \Psi_o)] \tag{4.84}$$

where
$$\phi - \Omega = \tan^{-1} \frac{L_x}{L_y} \tag{4.85}$$

$$\Psi_o = \tan^{-1} \frac{P_z L_y (L_x^2 + L_y^2 + L_z^2)^{1/2}}{P_x(L_x^2 + L_y^2) + P_z L_x L_z} \tag{4.86}$$

and

$$\beta = \tan^{-1} \frac{(L_x^2 + L_y^2)^{1/2}}{L_x} . \tag{4.87}$$

The Two Dimensional Kepler Problem in Parabolic Coordinates

In parabolic coordinates, defined by

$$x = \mu\ \nu \tag{4.88}$$

$$y = (\mu^2 - \nu^2)/2 \tag{4.89}$$

and where $m = e = 1$, the Kepler Hamiltonian is

$$H = \frac{P_\mu^2 + P_\nu^2 - 4}{2(\mu^2 + \nu^2)} . \tag{4.90}$$

From the Hamilton-Jacobi equation

$$P_\mu = (2W\mu^2 + 2\alpha_1)^{1/2} \tag{4.91}$$

$$P_\nu = (2W\nu^2 + 2\alpha_2)^{1/2} \tag{4.92}$$

where

$$\alpha_1 + \alpha_2 = 2. \tag{4.93}$$

The action variables defined by

$$J_i = \oint p_i dq_i \tag{4.94}$$

give

$$J_\mu = - \frac{2\pi\alpha_1}{\sqrt{-2W}} \tag{4.95}$$

and
$$J_\nu = -\frac{2\pi\alpha_2}{\sqrt{-2W}} \; . \tag{4.96}$$

Solving Eqs. (4.91), (4.92), (4.95) and (4.96) for α_1 and α_2 and equating the solutions

$$-\frac{\sqrt{-2W}}{2\pi} J_\mu = (P_\mu^2 - 2W\mu^2)/2 \tag{4.97}$$

and

$$-\frac{\sqrt{-2W}}{2\pi} J_\nu = (P_\nu^2 - 2W\nu^2) \; . \tag{4.98}$$

Hence

$$J_\mu = -\pi\sqrt{-2W}\,(\mu^2 + \frac{P_\mu^2}{(-2W)}) \tag{4.99}$$

$$J_\nu = -\pi\sqrt{-2W}\,(\nu^2 + \frac{P_\nu^2}{(-2W)}) \; . \tag{4.100}$$

The form of these latter two equations suggests that two new variables be defined

$$a = \mu + \frac{i}{\sqrt{-2W}} P_\mu \tag{4.101}$$

$$b = \nu + \frac{i}{\sqrt{-2W}} P_\nu \; , \tag{4.102}$$

so that

$$J_\mu = -\pi\sqrt{-2W}\; aa^* \tag{4.103}$$

$$J_\nu = -\pi\sqrt{-2W}\; bb^* \; . \tag{4.104}$$

Both a and b are very similar to the harmonic oscillator operators introduced in Chapter 2 and by analogy one would expect to find that quantities proportional to ab^* and a^*b are also constants of the motion. This is in fact the case. Taking real and imaginary parts of these quantities and normalizing, the constants are

$$K = -\frac{\sqrt{-2W}}{2}(ab^* + a^*b) \tag{4.105}$$

and

$$L = -i\frac{\sqrt{-2W}}{2}(ab^* - a^*b). \tag{4.106}$$

The third independent constant can be chosen to be the difference between J_μ and J_ν

$$D = \frac{\sqrt{-2W}}{2}(aa^* - bb^*) \; . \tag{4.107}$$

It is not difficult to verify that these constants satisfy the commutation relations of the generators of SU_2 or R_3 i.e.,

$$\{K, L\} = -2D \tag{4.108}$$

$$\{L, D\} = -2K \tag{4.109}$$

$$\{D, K\} = -2L \; , \tag{4.110}$$

where

$$\{a^*, a\} = \{b^*, b\} = \frac{2i}{\sqrt{-2W}} \quad . \tag{4.111}$$

It should be noted that in this case the angle variables w_μ and w_ν were not used. This occurred only because a similarity was noted between the constants of the motion J_μ and J_ν and the constants of the motion for the harmonic oscillator. Implicit in the definitions of a and b are two variables, denoted β and γ, which are conjugate to J_μ J_ν but these are not the angle variables. These variables are

$$\beta = \frac{1}{2\pi}\tan^{-1}\frac{P_\mu}{\sqrt{-2W\mu}} \tag{4.112}$$

and

$$\gamma = \frac{1}{2\pi}\tan^{-1}\frac{P_\nu}{\sqrt{-2W\nu}} \quad . \tag{4.113}$$

It is easy to verify that these are conjugate to the action variables

for

$$\{\gamma,\ J_\nu\} = \{\beta,\ \mathcal{J}_\mu\} = 1 \tag{4.114}$$

and

$$\{\gamma,\ \mathcal{J}_\mu\} = \{\beta,\ J_\nu\} = 0\ . \tag{4.115}$$

It should not be surprising to find a set of coordinates conjugate to the action variables which are not the angle variables because for a given set of momenta the set of canonical coordinates is not unique. This occurs because one can always add a total time derivative to the Lagrangian which is a function of q, $\dot{q}$ and t.

The constants K, L, and D have remarkably simple interpretations. K and D are the x and y components of the vector $\overline{P} = \overline{R}\sqrt{-2W}$ where R is the Runge vector, while $\overline{L}$ is the angular momentum. This can be verified by substituting Eqs. (4.101) and (4.102) into Eqs. (4.105) through (4.107) and then transforming either to rectangular or polar coordinates.

CHAPTER 5

SUMMARY

First of all, the motivation of this dissertation was to find general principles under which symmetry groups of Hamiltonian systems may be found. One is guided in the search by the known facts that the isotropic harmonic oscillator and the Kepler problems have such symmetry groups, as has been shown, for example, by Saenz [11], who reduced each of these problems to force free motion on the surface of a hypersphere. This is the central idea behind the stereographic parameters of Laporte and Rainich [7]; indeed their guiding principle is that for every hypersphere there will be a projection associated with some potential, which will have accidental degeneracy on that account.

However, the ideas expressed in this dissertation are different and depend upon two stages of transformation. The first is to reexamine the harmonic oscillator, using another mapping, the Hopf mapping, which reduces the orbits to points. Rotations in the resulting space, corresponding to SU_n transformations of the original space, are readily verified to form a symmetry group. The exceptional role played by the harmonic oscillator in this regard is simply that its Hamiltonian is quadratic, and that the collection of all quadratic functions over the phase space forms a Lie algebra which operates on phase space in a particularly interesting manner. There is then quite generally an SU_n group commuting with any normal operator, not always formed from

quadratic operators except in the isotropic case, for which the normal operator is degenerate.

The second stage is to consider what Hamiltonians are equivalent to quadratic Hamiltonians by canonical transformations. In this regard one can say almost all are. Here, one reinterprets the Hamilton-Jacobi equation. If it is thought of as yielding force-free motion in some (generally non-euclidean) phase space, then a further transformation to polar coordinates will yield circular motion in the phase planes, which presumably implies a quadratic relation between the coordinate and conjugate momentum.

This relation is delicate on two accounts. First the relationship in one phase plane may depend upon separation constants arising from other phase planes, and thus irreconcilable conflicts may arise from inverting all the transformations to the Hamilton-Jacobi system. Secondly, with an eye toward quantum mechanical applications, the symmetry operators may well be transcendental functions of the coordinates and momenta, making it impossible to define the proper quantum mechanical functions of non-commuting operators.

The organization of the paper was as follows: In the first chapter consideration is given to the Lie algebra generated by quadratic functions over the phase space under the Poisson bracket operation, and its relation to different homogeneous function spaces also taken over the phase space. It is seen that the eigenfunctions of these quadratic operators in the phase space itself determine the structure of all the other homogeneous spaces; and in particular there is a simple criterion for finding a set

of operators which commute with the original operator under Poisson brackets. Namely, it is that they be constructed from products of phase space eigenfunctions belonging to positive and negative pairs of eigenvalues. These correspond to the "constants of the motion" when the operator is taken to be the Hamiltonian. It is also seen that such positive and negative pairs will exist on account of the general existence of a set of operators formed from time-reversal and coordinate momentum interchange operators. Even though the constants will form a linearly independent set there may exist functional relationships among them as one expects to find at most (2n - 1) time independent constants for a Hamiltonian with n degrees of freedom.

The constants commuting with a degenerate normal operator generate an SU_n group, while the quadratic functions themselves generate an Sp_n group (symplectic group in n dimensions). A non-normal operator generates a somewhat different group --- a semidirect product [41] of a rotation group and an Abelian group, while normal, non-degenerate operators generate an SU_n group, whose operators are generally not homogeneous functions, but suitable roots of the latter.

Secondly, consideration is given to quadratic Hamiltonians in considerable detail, the second chapter being devoted to isotropic and anisotropic harmonic oscillators, specifically in two and generally in n dimensions. For the isotropic oscillators a particular Hopf mapping yields a direct geometric interpretation of the constants through a canonical mapping. In the anisotropic case it is not directly a geometrical transformation, but nevertheless canonical coordinates and

momenta are found such that the momenta are n-1 of the generators of the SU_n group which generate rotations in the coordinate planes.

The third chapter concerns another specific class of quadratic Hamiltonians, which may be interpreted as belonging to a charged particle moving in a uniform magnetic field as well as a plane harmonic oscillator potential. The problem is of some interest in its own right, and among other things, it is learned how to transfer constants of the motion to a rotating coordinate system. In particular a rule is given for writing Hamiltonians with central potentials when a uniform magnetic field is added and a transformation is made to a rotating coordinate system with a frequency of rotation equal to the Larmor frequency. The generators of the symmetry group yield a set of intrinsic coordinates, which allow a clean separation of the magnetic field (or angular momentum) from the oscillatory motion; a transformation used by Goshen and Lipkin [24] in their heuristic treatment of rotational bands. In addition there emerged a relation between the oscillator frequency, the Larmor frequency and a parameter, which relation yields bounded closed orbits if the parameter is a rational number. Finally, by taking high field and low field limits, one obtains Larmor precession of the harmonic oscillator orbits for a weak field, and the drift of the cyclotron orbits along the harmonic oscillator trajectories in the high field case. Viewed from a rotating coordinate system with the Larmor frequency all orbits become pure harmonic oscillator orbits. Of application to general problems of motion in a magnetic field and adiabatic constants, this problem is soluble analytically, so that the magnetic

field and potential field's effects may be seen clearly. A careful treatment is given of the known constants of the motion of the cyclotron problem, known from the work of Johnson and Lippmann [42]. Particular attention is paid to the influence of the vector potential on the obvious geometric symmetry, and it is seen that canonical rotations and geometrical rotations yield two entirely different types of symmetry and constants of the motion.

Although the specific form of the Hamiltonian has a particularly rich variety of interpretations as a physical problem regarding the motion of a charged particle, mathematically it is essentially nothing but a (possibly anisotropic) harmonic oscillator, a fact which reduces the discovery of the symmetry group to a previous case, and renders the analytic solution of the problem trivial. In general the orbit is a Lissajous figure and the constants of the motion have their interpretation as operators which move the boundaries of the figure or change the phase of the orbit in various ways, preserving the sum of the squares of its semiaxes --- upon which the energy of the orbit solely depends.

In the fourth chapter consideration is given to a non-harmonic problem, the Kepler problem, which is well known to be accidentally degenerate, and equivalent in the bound orbits at least to the harmonic oscillators. As a new result it is found that there is an SU_3 symmetry group, in addition to the known R_4 symmetry group, the additional constants defining the line of nodes. In any event the generators of the two groups are functions of one another and not subsets. Thus, one has a general functional relation between the generators of these two groups.

As a general rule one can say that any Hamiltonian which is separable will possess an SU_n symmetry even though the frequencies in each angle coordinate may not be equal.

Since it is of considerable importance for the further application of this technique, the relation of the SU_3 group arising from different separations of the Hamilton-Jacobi equation in polar and parabolic coordinates is studied. It is found that considerable attention has to be paid to the problem of finding rational constants of the motion.

This work offers many prospects for continuation. There are still many potentials, for which hope of finding rational constants seems slim, but which nevertheless ought to be investigated. Among these are the Stark effect (Redmond recently found a new constant [45]), the problem of two fixed attracting centers [46], the motion of a charged particle in the field of a magnetic dipole, as well as the relativistic Kepler problem which Biedenharn has investigated recently [27, 28]. In addition, the relations uncovered between the orthogonal groups and unitary groups might lead to some interesting formulas for the higher dimensional groups, but a point of particular interest is the appearance of the angular momentum operators as a subgroup of SU_3. Since they depend on two of the three coordinates of phase space, one can exploit the symmetry of SU_3 as a permutation of its generators to obtain SU_2 groups involving the remaining coordinate, and hence the radial action. In quantum mechanical terms, this yields another viewpoint toward the known fact that coulomb integrals behave as Wigner coefficients [47].

In conclusion it should be pointed out that due to the nonlinearity of the various relations found, their extension to quantum mechanical operators will require exceptionally careful scrutiny.

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BIOGRAPHY

Victor August Dulock, Jr. was born February 26, 1939, at Waco, Texas. He was graduated from St. Thomas High School, Houston, Texas in June, 1956. In June, 1960 he received the degree of Bachelor of Arts from the University of St. Thomas in Houston, Texas. From September, 1960 until the present time he has pursued his work toward the degree of Doctor of Philosophy in the Department of Physics at the University of Florida. During his graduate career, he worked as a graduate assistant and interim instructor in the Department of Physics.

Mr. Dulock is married to the former Maryann Welhoelter of Houston, Texas and is the father of three children. He is a member of the American Physical Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 18, 1964

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